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Periodic Patterns, Linear Instability, Symplectic Structure and Mean-Flow Dynamics for Three-Dimensional Surface Waves

Thomas J. Bridges

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Periodic patterns, linear instability, symplectic structure and mean-flow dynamics for three-dimensional surface waves

By Thomas J. Bridges

Department of Mathematical and Computing Sciences, University of Surrey, Guildford GU2 5XH, Surrey, UK

Contents

Space and time periodic waves at the two-dimensional surface of an irrotational inviscid fluid of finite depth are considered. The governing equations are shown to have a new formulation as a generalized Hamiltonian system on a multisymplectic structure where there is a distinct symplectic operator corresponding to each unbounded space direction and time. The wave-generated mean flow in this framework has an interesting characterization as drift along a group orbit. The theory has interesting connections with, and generalizations of, the concepts of action, action flux, pseudofrequency and pseudowavenumber of the Whitham theory. The multisymplectic structure and novel characterization of mean flow lead to a new constrained variational principle for all space and time periodic patterns on the surface of a finite-depth fluid. With the additional structure of the equations, it is possible to give a direct formulation of the linear stability problem for three-dimensional travelling waves. The linear instability theory is valid for waves of arbitrary amplitude. For weakly nonlinear waves the linear instability criterion is shown to agree exactly with the previous results of Benney–Roskes, Hayes, Davey–Stewartson and Djordjevic–Redekopp obtained using modulation equations. Generalizations of the instability theory to study all periodic patterns on the ocean surface are also discussed.

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1. Introduction

In this paper various questions on the formation and linear stability of finite-amplitude multidimensional periodic patterns on the ocean surface, particularly when the depth is finite and mean-flow dynamics are important, are considered. The organizing centre for the analysis is a generalized Hamiltonian structure, multisymplectic structure (cf. Bridges 1996*b*), where a distinct symplectic operator is assigned for each unbounded direction and time. This structure is a decomposition of the usual Hamiltonian structure for water waves and leads to new and more precise results on wave-meanflow coupling and the linear instability of two- and three-dimensional travelling waves.

The present understanding of the instability of a uniformly travelling train of Stokes waves to sideband perturbations had its beginnings with the work of Benjamin-Feir (1967), Benjamin (1967), Lighthill (1965, 1967) and Whitham (1965, 1967). Initially, it was found both experimentally and theoretically that the Stokes wavetrain, travelling in one space direction, on a fluid of infinite depth, is linearly unstable at low amplitudes (cf. Benjamin-Feir 1967; Whitham 1967) but restabilizes at finite amplitude (Lighthill 1965, 1967). In finite depth, the theory is more difficult due to wave-induced mean-flow effects and Benjamin (1967) and Whitham (1967), using different methods, showed that, for a Stokes wave of fixed wavelength and sufficiently small amplitude, the sideband perturbations stabilize when the depth is sufficiently small. In the recent work of Bridges & Mielke (1996), a rigorous proof of the instability of the Stokes wavetrain has been obtained.

With the addition of three dimensionality, i.e. enlargement of the class of perturbations to include spanwise—normal to the direction of propagation of the basic state—perturbations, new instabilities were revealed. Zakharov (1968), using a Hamiltonian formulation, derived a nonlinear Schrödinger equation for modulated Stokes waves on an infinite depth fluid. Benney & Roskes (1969) derived a set of modulation equations, using the method of multiple scales, for Stokes waves travelling in a finite-depth fluid, showing that oblique instabilities could develop in finite depth. In Hayes (1973), the Whitham theory was generalized to higher dimension and further results on the instability of weakly nonlinear Stokes waves were presented. Davey & Stewartson (1974) showed that modulated Stokes waves, in 2(horizontal) + 1(vertical) space dimensions, travelling in finite depth satisfy a particular coupled set of modulation equations now known as the 'Davey–Stewartson equations', where the coupling is due to the wave-generated mean flow. The results obtained by Davey & Stewartson for instability of a monochromatic wavetrain agree exactly with the results of Benney & Roskes (1969) and Hayes (1973) (cf. Davey & Stewartson 1974, p. 109). The Benney–Roskes and Davey–Stewartson theory was extended by Djordjevic & Redekopp (1977) to include the effect of surface tension, revealing new effects, particularly a long-wave–short-wave resonance. The importance of three dimensionality in wave instabilities has also been confirmed experimentally (cf. Su 1982; Su *et al.* 1982).

Beginning with the work of Longuet-Higgins (1978) and McLean *et al.* (1981), significant new results on two- and three-dimensional instabilities were obtained numerically. Numerical methods for studying instabilities have three important advantages: the linear stability exponents (the spectrum) are computed directly; basic states of arbitrary amplitude can be studied; and a more complete perturbation class (both long-wave and short-wave perturbations) can be considered. A review of the numerical studies up to 1985, with particular emphasis on three-dimensional instabilities of waves on deep water, is given by Saffman & Yuen (1985). The above numerical results are for a basic state that is two dimensional (i.e. no spanwise variation of the basic state), whereas the perturbation class is three dimensional. Recently, the instability of three-dimensional basic states, particularly short-crested Stokes waves, has been studied numerically (cf. Ioualalen & Kharif 1994) leading to new types of oblique instabilities. Numerical methods will not be considered explicitly in this paper, although in § 6 there is some discussion of how the results of this paper can contribute to the numerical computation of wave instabilities.

The advantage of using modulation equations such as the Benney–Roskes theory, Davey–Stewartson equation or the nonlinear Schrödinger equation is that the methods are equally applicable in the conservative and non-conservative case (cf. Stuart & DiPrima 1978; Holmes 1985). On the other hand, of the theoretical methods proposed for analyzing the two- and three-dimensional instabilities of travelling water waves (Fourier method, Whitham's averaged Lagrangian and modulation equations, Benney–Roskes theory, Zakharov equation, nonlinear Schrödinger equation and the Davey–Stewartson equations), the only theory which is valid for basic states of arbitrary amplitude is the Whitham theory.

In the absence of mean-flow effects, the Whitham theory, for the modulation of a monochromatic wave travelling in one space dimension, can be summarized as follows (cf. Whitham 1974). Suppose the problem is governed by a Lagrangian function, evaluated at the monochromatic wave $\eta(\theta)$ where θ is a phase function. After averaging, the Lagrangian can be reduced to

$$\mathcal{L}(\omega, k), \quad \text{where} \quad \omega = -\theta_t \quad \text{and} \quad k = \theta_x.$$

Whitham introduces two functionals

$$\mathcal{A} = \frac{\partial \mathcal{L}}{\partial \omega} \quad \text{and} \quad \mathcal{B} = -\frac{\partial \mathcal{L}}{\partial k}$$

and then derives the Whitham modulation equations

$$\frac{\partial \mathcal{A}}{\partial t} + \frac{\partial \mathcal{B}}{\partial x} = 0 \quad \text{and} \quad \frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0. \tag{1.1}$$

The first equation of (1.1) is called the 'conservation of wave action' (cf. Whitham 1965, 1967; Hayes 1970) and the second is a kinematic condition arising from the definition of the phase function θ. It is remarkable that the two simple equations (1.1) are sufficient to predict instability for waves of arbitrary amplitude. Some partial rigorous results on the validity of these equations are given in Bridges (1996*b*, §§ 4 and 5). The Whitham modulation equations will not be used in the present work as we will show shortly that the linear instability problem can be attacked directly when the complete set of governing equations is cast into a generalized Hamiltonian form. However, the information provided by equations (1.1) is instructive.

Treating $\mathcal{A}$ and $\mathcal{B}$ as functions of ω and k, system (1.1) becomes

$$\begin{bmatrix} \mathcal{A}_\omega & \mathcal{A}_k \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \omega \\ k \end{pmatrix}_t + \begin{bmatrix} \mathcal{B}_\omega & \mathcal{B}_k \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \omega \\ k \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.2}$$

If one lets

$$\omega = \omega_0 + \widehat{\omega} \mathrm{e}^{\mathrm{i}(\alpha x - \Omega t)} \quad \text{and} \quad k = k_0 + \widehat{k} \mathrm{e}^{\mathrm{i}(\alpha x - \Omega t)},$$

where (ω_0, k_0) are representative of a basic state, then substitution into (1.2) results in the following algebraic equation for the stability exponent Ω:

$$\mathcal{A}_\omega^0 \Omega^2 + (\mathcal{A}_k^0 - \mathcal{B}_\omega^0)\Omega\alpha - \mathcal{B}_k^0 \alpha^2 = 0. \tag{1.3}$$

It was first recognized by Lighthill (1965), using the Whitham formulation, that if $\mathcal{L}(\omega, k)$ (or $-\mathcal{L}$), evaluated on a basic state, is a convex function of (ω, k) or

$$\Delta_{\mathcal{L}} = \det \begin{bmatrix} \mathcal{L}_{\omega\omega}^0 & \mathcal{L}_{\omega k}^0 \\ \mathcal{L}_{k\omega}^0 & \mathcal{L}_{kk}^0 \end{bmatrix} > 0, \tag{1.4}$$

then the basic state is linearly unstable. This was argued on the grounds that solving (1.3) and using the identities $\mathcal{A} = \mathcal{L}_\omega$ and $\mathcal{B} = -\mathcal{L}_k$ leads to

$$\Omega = \left\{ \frac{\mathcal{L}_{\omega k}^0 \pm \sqrt{-\Delta_{\mathcal{L}}}}{\mathcal{L}_{\omega\omega}^0} \right\} \alpha.$$

Therefore, if $\Delta_{\mathcal{L}} > 0$ and $\alpha \neq 0$, there exists a solution to equation (1.2), linearized about the basic state, that grows exponentially in time. The criterion (1.4) is interesting because is requires only information that is readily obtained from the existing basic state. However, what is not obvious is that an unstable solution of (1.2) should imply an unstable solution of the original equations from which (1.2) is derived.

A difficulty with the above instability criterion and the Whitham modulation equations, in general, is that the connection with the physical problem is not clear. When $\Delta_{\mathcal{L}} > 0$, one would like to construct, explicitly, or approximately, an unstable eigenfunction and an expression for the linear stability exponent for the original physical problem. This turns out to be possible by treating the problem in a generalized Hamiltonian framework rather than a Lagrangian framework. The importance of a Hamiltonian framework is that there is additional structure, given by the symplectic operators, in the governing equations. In the recent work of Bridges (1996*b*), a theory is presented that establishes the connection between the above instability criterion and the original physical problem. In fact, for a general class of problems, a rigorous proof can be given which also carries over to three-dimensional waves (cf. Bridges 1996*b*, § 4). The idea is to work in the Hamiltonian setting rather than the Lagrangian setting. However, the usual Hamiltonian setting, obtained by eliminating the time derivatives in the Lagrangian via, for example, a Legendre transformation, is insufficient. To obtain sufficient structural information, one must Legendre transform, independently, *each* unbounded space direction and time. In this way one obtains significant information about the structure of the functionals $\mathcal{A}$, the action, and $\mathcal{B}$, the action 'flux'. This theory, termed multisymplectic structure, since there is a family of symplectic operators, leads to a number of new results on the dynamics and instability of waves (cf. Bridges 1996*b*).

In the present paper, it is shown that the multisymplectic structure formulation of the governing equations (see equation (1.5)) leads to new results on the formal existence and formal instability of three-dimensional space and time periodic patterns on a finite-depth fluid. No attempt will be made in the present work to give a rigorous mathematical proof of the results. The purpose is to present a new framework for analysing periodic patterns on the ocean surface, particularly when the depth is finite.

In § 2 the governing equations for inviscid irrotational three-dimensional water

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waves are considered and it is shown that there exists a vector-valued set of dependent variables, denoted Z, such that the governing equations take the geometric form

$$\boldsymbol{M}Z_t + \boldsymbol{K}(\boldsymbol{u})Z_x + \boldsymbol{L}(\boldsymbol{v})Z_y = \nabla S(Z), \tag{1.5}$$

where $(\boldsymbol{M}, \boldsymbol{K}(\boldsymbol{u}), \boldsymbol{L}(\boldsymbol{v}))$ are a family of symplectic operators (each is skew-symmetric and closed, i.e. each term $(\boldsymbol{M}Z_t, \boldsymbol{K}(\boldsymbol{u})Z_x, \boldsymbol{L}(\boldsymbol{v})Z_y)$ is the gradient of a functional) and $\nabla S(Z)$ is the gradient of a functional $S(Z)$ (cf. equation (2.9) and neighbouring text). In fact, $(\boldsymbol{M}Z_t, \boldsymbol{K}(\boldsymbol{u})Z_x, \boldsymbol{L}(\boldsymbol{v})Z_y)$ are the gradients of functionals that are related to the action and action flux; the above family of skew-symmetric operators contain the structural information about the action functionals needed for the linear stability analysis and other properties of wave propagation.

When the depth is finite, an important part of wave dynamics is the mean flow generated by the waves. In the Whitham formulation the mean-flow dynamics are modelled by introducing a pseudofrequency and pseudowavenumber coupled to a conservation law (cf. Whitham 1967, 1974; Hayes 1973). In the present paper a new characterization of mean flow is obtained in terms of the flow along a group orbit, where the symmetry group is associated with constant perturbation of the velocity potential.

A fundamental symmetry of the water-wave problem is perturbation of the potential. The action of this symmetry group on functions of the type in (1.5), denoted Γ, is

$$\Gamma \cdot Z = Z + \lambda V, \quad \forall \lambda \in \boldsymbol{R}, \tag{1.6}$$

where V is a constant vector (cf. equation (2.17)). Physically, the action (1.6) is $\phi \mapsto \phi + \lambda$ where ϕ is the velocity potential of the fluid. This seemingly trivial symmetry is of fundamental importance. In § 2 the mean flow will be characterized as flow along the λ-orbit; that is, the general solution is decomposed into two components

$$Z(x, y, z, t) + \lambda(x, y, t)V,$$

where the first part is the wave and the second is the mean flow (cf. equation (2.23)). Note that the mean flow is precisely in the direction of the group action. In fact, the concept of mean flow will be defined in precisely this way. It should also be noted that the mean flows under consideration are irrotational and therefore mean flows such as shear currents are excluded.

The decomposition of the wave dynamics into a wave component and mean-flow component, as well as the multisymplectic structure formulation, leads to a new variational principle for all space and time periodic patterns on a finite-depth fluid. In § 3 it is shown that, when the mean flow is of the form $\lambda(x, y, t) = \gamma t + m_1 x + m_2 y$, where γ, m_1 and m_2 are real numbers, to be determined, space and time periodic waves correspond to critical points of the Hamiltonian functional $S(Z)$ in (1.5), restricted to a family of six constraint sets. Let

$$\mathcal{C}(I) = \{Z : \overline{A} = I_1,\ \overline{B}_1 = I_2,\ \overline{B}_2 = I_3,\ \overline{P} = I_4,\ \overline{Q}_1 = I_5,\ \overline{Q}_2 = I_6\},$$

where the overbar indicates averaging over $\boldsymbol{T}^3$ and $I_1, \ldots, I_6$ are specified real numbers. The functionals (A, B_1, B_2) are action densities corresponding to the (t, x, y) directions, respectively, P is the mass density, and $\boldsymbol{Q} = (Q_1, Q_2)$ are the components of the mass-flux density. Let $\mathcal{S}$ be the S-Hamiltonian functional in (1.5) averaged over $\boldsymbol{T}^3$. Then, in § 3 it is shown that spacetime periodic waves, with coupled mean flow dynamics of the specified form, correspond to

$$\mathrm{crit}\{\mathcal{S} : Z \in \mathcal{C}(I)\}. \tag{1.7}$$

Introducing Lagrange multipliers $(\omega, \boldsymbol{k}, \gamma, \boldsymbol{m})$, where $\boldsymbol{k} = (k_1, k_2)$ and $\boldsymbol{m} = (m_1, m_2)$, the density of the functional for the Lagrange necessary condition associated with the constrained variational formulation (1.7) is

$$\mathcal{F}(Z, \omega, \boldsymbol{k}, \gamma, \boldsymbol{m}) = S(Z) - \omega A(Z) - \boldsymbol{k} \cdot \boldsymbol{B}(Z) - \gamma P(Z) - \boldsymbol{m} \cdot \boldsymbol{Q}(Z),$$

where $\boldsymbol{B} = (B_1, B_2)$ and $\boldsymbol{Q} = (Q_1, Q_2)$. The above variational principle leads to an interesting organization of the parameter structure of the problem. There are two sets of parameters: the values for the level sets $I_1, \cdots, I_6$, which are specified, and the Lagrange multipliers $(\omega, \boldsymbol{k}, \gamma, \boldsymbol{m})$, which are to be determined. The variational principle (1.7) is said to be non-degenerate when

$$\det(\boldsymbol{D}) \neq 0 \quad \text{where} \quad \boldsymbol{D} = \begin{bmatrix} \dfrac{\partial \overline{A}}{\partial \omega} & \dfrac{\partial \overline{A}}{\partial k_1} & \dfrac{\partial \overline{A}}{\partial k_2} & \dfrac{\partial \overline{A}}{\partial \gamma} & \dfrac{\partial \overline{A}}{\partial m_1} & \dfrac{\partial \overline{A}}{\partial m_2} \\ \dfrac{\partial \overline{B_1}}{\partial \omega} & \dfrac{\partial \overline{B_1}}{\partial k_1} & \dfrac{\partial \overline{B_1}}{\partial k_2} & \dfrac{\partial \overline{B_1}}{\partial \gamma} & \dfrac{\partial \overline{B_1}}{\partial m_1} & \dfrac{\partial \overline{B_1}}{\partial m_2} \\ \dfrac{\partial \overline{B_2}}{\partial \omega} & \dfrac{\partial \overline{B_2}}{\partial k_1} & \dfrac{\partial \overline{B_2}}{\partial k_2} & \dfrac{\partial \overline{B_2}}{\partial \gamma} & \dfrac{\partial \overline{B_2}}{\partial m_1} & \dfrac{\partial \overline{B_2}}{\partial m_2} \\ \dfrac{\partial \overline{P}}{\partial \omega} & \dfrac{\partial \overline{P}}{\partial k_1} & \dfrac{\partial \overline{P}}{\partial k_2} & \dfrac{\partial \overline{P}}{\partial \gamma} & \dfrac{\partial \overline{P}}{\partial m_1} & \dfrac{\partial \overline{P}}{\partial m_2} \\ \dfrac{\partial \overline{Q_1}}{\partial \omega} & \dfrac{\partial \overline{Q_1}}{\partial k_1} & \dfrac{\partial \overline{Q_1}}{\partial k_2} & \dfrac{\partial \overline{Q_1}}{\partial \gamma} & \dfrac{\partial \overline{Q_1}}{\partial m_1} & \dfrac{\partial \overline{Q_1}}{\partial m_2} \\ \dfrac{\partial \overline{Q_2}}{\partial \omega} & \dfrac{\partial \overline{Q_2}}{\partial k_1} & \dfrac{\partial \overline{Q_2}}{\partial k_2} & \dfrac{\partial \overline{Q_2}}{\partial \gamma} & \dfrac{\partial \overline{Q_2}}{\partial m_1} & \dfrac{\partial \overline{Q_2}}{\partial m_2} \end{bmatrix}. \tag{1.8}$$

Moreover, when $|\boldsymbol{D}| \neq 0$, $\boldsymbol{D}^{-1} = \mathrm{Hess}_I(\mathcal{S})$ (cf. equation (3.11)) where $\mathrm{Hess}_I(\mathcal{S})$ is the 6×6 matrix of second partial derivatives of $\mathcal{S}$ with respect to I. In fact, the identity $\boldsymbol{D}^{-1} = \mathrm{Hess}_I(\mathcal{S})$ is a remarkable consequence of the variational principle (1.7). In § 4 it will be shown that the elements in the matrix $\boldsymbol{D}$ can be related to the exact linear stability exponents for the full problem.

In § 4, the linear stability problem for periodic travelling waves, with coupled mean-flow effects of the form specified in § 3, is formulated and a criterion for linear instability is derived and characterized in terms of the elements of the matrix $\boldsymbol{D}$ in (1.8), thereby connecting the constrained variational principle of § 3 with the linear stability exponents.

In the governing equations (1.5), let $Z = \widehat{Z} + \widehat{U}$ where $\widehat{Z}$ represents the basic state. Then substitution into (1.5) and linearization about the basic state results in a linear partial differential equation with periodic coefficients. A particular class of perturbations of the form

$$\widehat{U} = \mathrm{Re}[U \mathrm{e}^{\mathrm{i}(p_1 \tilde{x} + p_2 \tilde{y} - \Omega \tilde{t})}]$$

is studied, where $\boldsymbol{p} = (p_1, p_2) \in \boldsymbol{R}^2$, $\Omega \in \boldsymbol{C}$ and U is a vector-valued eigenfunction. If

$\mathrm{Im}(\Omega) > 0$, it is clear that such a perturbation will be linearly unstable and because of symmetry this is equivalent to $\mathrm{Im}(\Omega) \neq 0$. In § 4 it is shown that when $|\boldsymbol{p}|$ and $|\Omega|$ are sufficiently small, the exponents satisfy the 'dispersion relation'

$$\Upsilon(\Omega, \boldsymbol{p}) = \det[\boldsymbol{N}_2(\Omega, \boldsymbol{p})] + R_5(\Omega, \boldsymbol{p}), \tag{1.9}$$

where R_5 contains terms of the form $\Omega^i p_1^j p_2^k$ with $i + j + k \geqslant 5$ and $\boldsymbol{N}_2(\Omega, \boldsymbol{p})$ is a 2×2 matrix and a homogeneous quadratic in $(\Omega, \boldsymbol{p})$; in particular,

$$\boldsymbol{N}_2(\Omega, \boldsymbol{p}) = \boldsymbol{N}_{21}\Omega^2 + \boldsymbol{N}_{22}(\boldsymbol{p})\Omega + \boldsymbol{N}_{23}(\boldsymbol{p}), \tag{1.10}$$

with

$$\left.\begin{aligned}
\boldsymbol{N}_{21} &= \begin{pmatrix} \dfrac{\partial \overline{A}}{\partial \omega} & -\dfrac{\partial \overline{A}}{\partial \gamma} \\[2ex] -\dfrac{\partial \overline{P}}{\partial \omega} & \dfrac{\partial \overline{P}}{\partial \gamma} \end{pmatrix}, \\
\boldsymbol{N}_{22}(\boldsymbol{p}) &= \sum_{j=1}^{2} p_j \begin{pmatrix} 2\dfrac{\partial \overline{A}}{\partial k_j} & \left(\dfrac{\partial \overline{A}}{\partial \beta_j} - \dfrac{\partial \overline{B}_j}{\partial \gamma}\right) \\[2ex] \left(\dfrac{\partial \overline{Q}_j}{\partial \omega} - \dfrac{\partial \overline{P}}{\partial k_j}\right) & -2\dfrac{\partial \overline{P}}{\partial \beta_j} \end{pmatrix}, \\
\boldsymbol{N}_{23}(\boldsymbol{p}) &= \sum_{i,j=1}^{2} p_i p_j \begin{pmatrix} \dfrac{\partial \overline{B}_i}{\partial k_j} & \dfrac{\partial \overline{B}_i}{\partial \beta_j} \\[2ex] \dfrac{\partial \overline{Q}_i}{\partial k_j} & \dfrac{\partial \overline{Q}_i}{\partial \beta_j} \end{pmatrix}.
\end{aligned}\right\} \tag{1.11}$$

Note that every entry of the matrix $\boldsymbol{D}$ in (1.8) appears in one of the matrices in (1.11), but they are organized in the way they contribute to the linear stability problem. From the theory of § 4 it follows that if there exists a root Ω of $\det[\boldsymbol{N}_2(\Omega, \boldsymbol{p})] = 0$ for $|\boldsymbol{p}|$ sufficiently small and $\mathrm{Im}(\Omega) \neq 0$, the basic state is linearly unstable. The proof of this result uses the multisymplectic structure and the characterization of the mean flow in a non-trivial way. An important result, fundamental to the analysis of the linear stability problem, is the ability to decompose the conservation laws, particularly the mass conservation law, according to the multisymplectic structure. In other words, the mass conservation law

$$\frac{\partial P}{\partial t} + \frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} = 0$$

has a decomposition according to the multisymplectic structure of the form

$$\boldsymbol{M}V = \nabla P(Z), \quad \boldsymbol{K}(\boldsymbol{u})V = \nabla Q_1(Z) \quad \text{and} \quad \boldsymbol{L}(\boldsymbol{v})V = \nabla Q_2(Z),$$

where V is the generating function for the group action (1.6), which in turn is related to the mean flow through the wave-meanflow decomposition and P, Q_1 and Q_2 are the components of the mass conservation law. The instability criterion of § 4 is also global; that is, there is no restriction on the amplitude of the basic state. Moreover, it is proved directly using the governing equations, without recourse to modulation equations or the conservation of wave action, and should therefore be useful for numerical computations as well.

In § 5 the theory is applied to a weakly nonlinear monochromatic Stokes wave travelling in two space dimensions with surface tension forces at the fluid surface taken into account. The variational principle of § 3 and the instability theory of § 4 are applied to the basic state. The instability results are in complete agreement with previous results in the literature (Benney & Roskes 1969; Hayes 1973; Davey & Stewartson 1974; Djordjevic & Redekopp 1977).

In the discussion in § 6 some generalizations of the instability theory are presented. Since the variational principle of § 3 is for all space and time periodic states with the specified form of the mean flow, it is of interest to develop an instability theory, along the lines of the theory in § 4, for more general classes of spacetime periodic states than travelling waves. Such a theory can lead to significant progress in the pattern formation question for the open ocean: what are all possible periodic patterns on the ocean surface and which are unstable and which, if any, are stable?

2. Governing equations, multisymplectic structure and the mean flow

The governing equations are formulated as follows. Let $(x, y) \in \boldsymbol{R}^2$ denote the horizontal coordinates and z the vertical coordinate. Denote by $\phi(x, y, z, t)$ the velocity potential for the inviscid irrotational and constant density fluid. The fluid is bounded below by a horizontal plane at $z = -h$ for some $h > 0$ (in Appendix A it is shown that h can be related to the uniform flow) and bounded above by the surface $z = \eta(x, y, t)$. In the interior of the fluid, the velocity potential satisfies Laplace's equation

$$\Delta\phi \stackrel{\text{def}}{=} \phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad \text{for} \quad -h < z < \eta(x, y, t) \tag{2.1}$$

and impermeability at the bottom requires

$$\phi_z = 0 \quad \text{at} \quad z = -h. \tag{2.2}$$

At the free surface the functions (ϕ, η) satisfy the kinematic and dynamic boundary conditions

$$\left.\begin{aligned} \eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0, \\ \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta - \sigma\left(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y}\right) = 0, \end{aligned}\right\} \quad \text{at } z = \eta(x, y, t), \tag{2.3}$$

where

$$w_1 \stackrel{\text{def}}{=} \frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_y^2}} \quad \text{and} \quad w_2 \stackrel{\text{def}}{=} \frac{\eta_y}{\sqrt{1 + \eta_x^2 + \eta_y^2}}. \tag{2.4}$$

The coefficients g and σ represent the gravitational and surface-tension coefficients, respectively.

The purpose of this section is to reformulate the governing equations (2.1)–(2.3) as a Hamiltonian system on a multisymplectic structure, following Bridges (1996*b*), and characterize the mean flow associated with the equations. Introduce the following set of new coordinates:

$$Z = \begin{pmatrix} \Phi \\ \eta \\ w_1 \\ w_2 \\ \phi \\ u \\ v \end{pmatrix}, \quad \text{where} \quad \begin{cases} \Phi = \phi|_{z=\eta}, \\ u = \phi_x, \\ v = \phi_y, \\ \boldsymbol{u} = u|_{z=\eta}, \\ \boldsymbol{v} = v|_{z=\eta}. \end{cases} \tag{2.5}$$

The functions (ϕ, η, w_1, w_2) are, for each (x, y, t), real-numbers, whereas (ϕ, u, v) are dependent on the cross section $z \in (-h, \eta)$. Using the fact that

$$\Phi_t = [\phi_t + \phi_z \eta_t]|_{z=\eta},$$

with similar relations for Φ_x and Φ_y, and the kinematic condition, leads to the identity

$$\Phi_t + \boldsymbol{u}\Phi_x + \boldsymbol{v}\Phi_y = [\phi_t + (\phi_x^2 + \phi_y^2 + \phi_z^2)]|_{z=\eta}. \tag{2.6}$$

The governing equations can be recast into the following form:

$$\left.\begin{aligned} -\eta_t - \boldsymbol{u}\eta_x - \boldsymbol{v}\eta_y &= -\phi_z|_{z=\eta}, \\ \Phi_t + \boldsymbol{u}\Phi_x + \boldsymbol{v}\Phi_y - \sigma\left(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y}\right) &= \tfrac{1}{2}(\boldsymbol{u}^2 + \boldsymbol{v}^2 + \phi_z^2)|_{z=\eta} - g\eta, \\ \sigma\eta_x = \sigma w_1/(1 - w_1^2 - w_2^2)^{1/2}, &\quad \sigma\eta_y = \sigma w_2/(1 - w_1^2 - w_2^2)^{1/2}, \\ -u_x - v_y = \phi_{zz}, &\quad \phi_x = u, \quad \phi_y = v. \end{aligned}\right\} \tag{2.7}$$

The advantage of the formulation (2.7) is that all x, y and t derivatives are on the left-hand side and, as will be demonstrated shortly, the right-hand side is the gradient of a functional. The first equation of (2.7) is the kinematic free surface condition in terms of the transformed coordinates; the third and fourth equations are identities corresponding to the inverse of (2.4); the fifth equation recovers Laplace's equation for ϕ when the definitions of u and v, given by the sixth and seventh equations, are substituted. Using (2.6) and the definitions in (2.5), one verifies that the second equation of (2.7) recovers the dynamic free surface condition.

The structure of the left-hand side of (2.7) can be further refined by separating the terms with x, y and t derivatives. Let

$$\boldsymbol{M} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{K}(\boldsymbol{u}) = \begin{pmatrix} 0 & -\boldsymbol{u} & 0 & 0 & 0 & 0 & 0 \\ \boldsymbol{u} & 0 & -\sigma & 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.8 a}$$

and

$$\boldsymbol{L}(\boldsymbol{v}) = \begin{pmatrix} 0 & -\boldsymbol{v} & 0 & 0 & 0 & 0 & 0 \\ \boldsymbol{v} & 0 & 0 & -\sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{2.8 b}$$

Then, (2.7) can be written in the form

$$\boldsymbol{M} Z_t + \boldsymbol{K}(\boldsymbol{u}) Z_x + \boldsymbol{L}(\boldsymbol{v}) Z_y = \nabla S(Z). \tag{2.9}$$

Using the definitions in (2.8) and writing out the left-hand side of (2.9), one recovers the left-hand side of (2.7). In order to demonstrate that the right-hand side of (2.7) is indeed the gradient of some functional S, we first introduce a suitable inner product. For vector-valued functions of the type (2.5), where the first four components are scalar-valued and the last three components are defined on the cross section $z \in (-h, \eta)$, the following inner product is defined:

$$\langle U, V \rangle_m = U_1 V_1 + U_2 V_2 + U_3 V_3 + U_4 V_4 + \int_{-h}^{\eta} (U_5 V_5 + U_6 V_6 + U_7 V_7)\, \mathrm{d}z. \tag{2.10}$$

Note that the inner product does not include integration over x, y or t. The subscript m on the inner product is an indication that the inner product is η-dependent.

Introduce the following functional:

$$S(Z) = \tfrac{1}{2} \int_{-h}^{\eta} (u^2 + v^2 - \phi_z^2)\, \mathrm{d}z - \tfrac{1}{2} g \eta^2 + \sigma \left(1 - \sqrt{1 - w_1^2 - w_2^2} \right). \tag{2.11}$$

Then with respect to the inner product (2.10), we find

$$\nabla S(Z) \stackrel{\text{def}}{=} \begin{pmatrix} \delta S / \delta \Phi \\ \delta S / \delta \eta \\ \delta S / \delta w_1 \\ \delta S / \delta w_2 \\ \delta S / \delta \phi \\ \delta S / \delta u \\ \delta S / \delta v \end{pmatrix} = \begin{pmatrix} -\phi_z|_{z=\eta} \\ \tfrac{1}{2}(\boldsymbol{u}^2 + \boldsymbol{v}^2 + \phi_z^2)|_{z=\eta} - g\eta \\ \sigma w_1 / \sqrt{1 - w_1^2 - w_2^2} \\ \sigma w_2 / \sqrt{1 - w_1^2 - w_2^2} \\ \phi_{zz} \\ u \\ v \end{pmatrix}, \tag{2.12}$$

which is precisely the right-hand side of (2.7). This completes the verification of (2.9) as a representation of the governing equations (2.1)–(2.3). One can also verify that, with respect to the inner product (2.10), the operators $\boldsymbol{M}$, $\boldsymbol{K}(\boldsymbol{u})$ and $\boldsymbol{L}(\boldsymbol{v})$ are skew-symmetric:

$$\langle \boldsymbol{M} U, V \rangle_m = -\langle U, \boldsymbol{M} V \rangle_m,$$
$$\langle \boldsymbol{K}(\boldsymbol{u}) U, V \rangle_m = -\langle U, \boldsymbol{K}(\boldsymbol{u}) V \rangle_m,$$
$$\langle \boldsymbol{L}(\boldsymbol{v}) U, V \rangle_m = -\langle U, \boldsymbol{L}(\boldsymbol{v}) V \rangle_m.$$

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Abstractly, the system (2.9) is a Hamiltonian system on a multisymplectic structure with Hamiltonian functional S and three symplectic forms $\omega^{(1)}$, $\omega^{(2)}$ and $\omega^{(3)}$. One can show that the three symplectic forms are exact (cf. Bridges 1996*b*, § 6) and therefore $\boldsymbol{M}Z_t$, $\boldsymbol{K}(\boldsymbol{u})Z_x$ and $\boldsymbol{L}(\boldsymbol{v})Z_y$ are the gradients of suitably defined action functionals (where the gradients in this case are defined with respect to an inner product that includes integration over x, y and t). The family of action functional densities is defined as follows:

$$\left.\begin{aligned} A(Z,Z_t) &= -\Phi\eta_t,\\ B_1(Z,Z_x) &= \int_{-h}^{\eta} u\phi_x\,\mathrm{d}z + \sigma w_1\eta_x,\\ B_2(Z,Z_y) &= \int_{-h}^{\eta} v\phi_y\,\mathrm{d}z + \sigma w_2\eta_y. \end{aligned}\right\} \tag{2.13}$$

Let

$$\langle\!\langle\cdot\,,\cdot\rangle\!\rangle_m = \int_{t_1}^{t_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2}\langle\cdot\,,\cdot\rangle_m\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t, \tag{2.14}$$

for a suitable cube in x, y and t. With fixed endpoint conditions at the edges of the cube in (2.14), gradients of A, B_1 and B_2, with respect to the inner product (2.14), are found to be

$$\nabla A(Z,Z_t) = \boldsymbol{M}Z_t,\quad \nabla B_1(Z,Z_x) = \boldsymbol{K}(\boldsymbol{u})Z_x,\quad \nabla B_2(Z,Z_y) = \boldsymbol{L}(\boldsymbol{v})Z_y. \tag{2.15}$$

In other words, the governing equation (2.9) is recovered by the first variation of the functional

$$\widehat{\mathcal{F}}(Z) = \int_{t_1}^{t_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2}\mathcal{F}(Z,Z_x,Z_y,Z_t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t, \tag{2.16 a}$$

where

$$\mathcal{F}(Z,Z_x,Z_y,Z_t) = A(Z,Z_t) + B_1(Z,Z_x) + B_2(Z,Z_y) - S(Z). \tag{2.16 b}$$

Stationarity of the functional $\widehat{\mathcal{F}}$ is a least-action principle for the system (2.9); a generalization of the classical least-action principle to the case where there is more that one action functional.

The functionals A, B_1 and B_2 are distinct action functionals for each space direction and time. They are fundamental elements of the structure of the equations and their gradients, in terms of the coordinates (2.5), generate the skew-symmetric operators $\boldsymbol{M}$, $\boldsymbol{K}(\boldsymbol{u})$ and $\boldsymbol{L}(\boldsymbol{v})$. When averaged over a one-dimensional phase, the values of the action functionals correspond to the Whitham–Hayes definition of action and action flux (cf. Whitham 1967, 1974; Hayes 1970). However, the use of suitable coordinates, in terms of which the action densities A, B_1 and B_2 are one-forms, reveals important structural information about the actions, particularly the family of skew-symmetric operators $\boldsymbol{M}$, $\boldsymbol{K}(\boldsymbol{u})$ and $\boldsymbol{L}(\boldsymbol{v})$ that are fundamental to the variational principle of § 3 and the geometric formulation of the linear stability problem in § 4.

The classical Hamiltonian formulation for the time-dependent water-wave problem (cf. Zakharov 1968; Broer 1974), when considered in terms of the set of variables Z, is recovered from (2.9) by taking

$$\mathcal{H}(Z) = \int_{y_1}^{y_2}\int_{x_1}^{x_2}[S(Z) - B_1(Z,Z_x) - B_2(Z,Z_y)]\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\boldsymbol{M} Z_t = \nabla \mathcal{H}(Z),$$

where $\mathcal{H}$ is the total energy. This form of the equations emphasizes the fact that the formulation in terms of a multisymplectic structure is analogous to decomposing the usual Hamiltonian functional in order to create spatial action functionals. Spatial Hamiltonian formulations for water waves (cf. Mielke 1991; Baesens & MacKay 1992; Bridges 1992, 1994) can also be recovered from (2.9). For example, letting $Z_y = 0$ and $\theta = x - ct$ reduces (2.9) to

$$\boldsymbol{J}(\widehat{\boldsymbol{u}}) Z_\theta = \nabla S(Z),$$

$$\boldsymbol{J}(\widehat{\boldsymbol{u}}) = \boldsymbol{K}(\widehat{\boldsymbol{u}}) - c\boldsymbol{M},$$

recovering the formulation in Bridges (1992, 1994).

For waves on the surface of a finite-depth fluid, the coupled mean-flow dynamics are of fundamental importance. In the present formulation the mean flow effects have an interesting characterization in terms of the flow along a group orbit. The group action Γ in question is a constant perturbation of the potential and is linked through Noether's theorem to the mass conservation law.

An action for the group Γ on functions of the form (2.5) is

$$\Gamma \cdot Z = Z + \lambda V, \quad \forall \lambda \in \boldsymbol{R}, \quad \text{with } V = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{2.17}$$

In terms of the physical coordinates the Γ-action is $\lambda \mapsto \phi + \lambda$ for all $\lambda \in \boldsymbol{R}$, where ϕ is the velocity potential. It is evident from the definition of $S(Z)$ in (2.11) that

$$S(Z + \lambda V) = S(Z) \quad \forall \lambda \in \boldsymbol{R}, \tag{2.18}$$

i.e. $S(Z)$ is Γ-invariant.

There is a connection, via Noether's theorem, between the Γ-symmetry and the mass conservation law. With the decomposition of the governing equations in terms of the multisymplectic structure, we obtain an interesting decomposition of the mass conservation law.

The mass conservation law for water waves on a fluid of unit density is

$$\frac{\partial P}{\partial t} + \frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} = 0, \tag{2.19}$$

where

$$P(Z) = \eta, \quad Q_1(Z) = \int_{-h}^{\eta} u \, \mathrm{d}z \quad \text{and} \quad Q_2(Z) = \int_{-h}^{\eta} v \, \mathrm{d}z. \tag{2.20}$$

Using the inner product $\langle \cdot , \cdot \rangle_m$ in (2.10), the gradients of the component functionals of the above conservation law are

$$\nabla P(Z) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \nabla Q_1(Z) = \begin{pmatrix} 0 \\ \boldsymbol{u} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \nabla Q_2(Z) = \begin{pmatrix} 0 \\ \boldsymbol{v} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.21}$$

Using the family of skew-symmetric operators $\boldsymbol{M}$, $\boldsymbol{K}(\boldsymbol{u})$ and $\boldsymbol{L}(\boldsymbol{v})$, the components of the conservation law (2.19) are related to the action of Γ through the following identities:

$$\boldsymbol{M}V = \nabla P(Z), \quad \boldsymbol{K}(\boldsymbol{u})V = \nabla Q_1(Z) \quad \text{and} \quad \boldsymbol{L}(\boldsymbol{v})V = \nabla Q_2(Z). \tag{2.22}$$

The identities (2.22) are easily verified using the definitions of $\boldsymbol{M}$, $\boldsymbol{K}(\boldsymbol{u})$, $\boldsymbol{L}(\boldsymbol{v})$ and V and the gradients (2.21). In fact the identities (2.22) imply (2.19) in the following way (cf. Bridges 1996*b*, Appendix C). Differentiating (2.18) with respect to λ and setting $\lambda = 0$ results in

$$\begin{aligned} 0 = -\frac{\mathrm{d}}{\mathrm{d}\lambda} S(Z+\lambda V)\Big|_{\lambda=0} &= -\langle \nabla S(Z), V\rangle_m \\ &= -\langle \boldsymbol{M}Z_t + \boldsymbol{K}(\boldsymbol{u})Z_x + \boldsymbol{L}(\boldsymbol{v})Z_y, V\rangle_m \\ &= \langle Z_t, \boldsymbol{M}V\rangle_m + \langle Z_x, \boldsymbol{K}(\boldsymbol{u})V\rangle_m + \langle Z_y, \boldsymbol{L}(\boldsymbol{v})V\rangle_m \\ &= \langle Z_t, \nabla P\rangle_m + \langle Z_x, \nabla Q_1\rangle_m + \langle Z_y, \nabla Q_2\rangle_m \\ &= \frac{\partial P}{\partial t} + \frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y}, \end{aligned}$$

where the skew-symmetry of $\boldsymbol{M}$, $\boldsymbol{K}(\boldsymbol{u})$ and $\boldsymbol{L}(\boldsymbol{v})$, the equations (2.9) and the identities (2.22) have been used.

In order to characterize the mean flow as drift along the Γ-group orbit, we let λ depend on x, y, t and decompose solutions of (2.9) into the form

$$Z + \lambda(x, y, t)V. \tag{2.23}$$

Substitution of (2.23) into (2.9) results in

$$\boldsymbol{M}(Z+\lambda V)_t + \boldsymbol{K}(\boldsymbol{u})(Z+\lambda V)_x + \boldsymbol{L}(\boldsymbol{v})(Z+\lambda V)_y = \nabla S(Z+\lambda V).$$

But since $S(Z+\lambda V) = S(Z)$, this expression reduces to

$$\begin{aligned} \boldsymbol{M}Z_t + \boldsymbol{K}(\boldsymbol{u})Z_x + \boldsymbol{L}(\boldsymbol{v})Z_y &= \nabla S(Z) - \lambda_t \boldsymbol{M}V - \lambda_x \boldsymbol{K}(\boldsymbol{u})V - \lambda_y \boldsymbol{L}(\boldsymbol{v})V \\ &= \nabla S(Z) - \lambda_t \nabla P(Z) - \lambda_x \nabla Q_1(Z) - \lambda_y, \nabla Q_2(Z), \end{aligned} \tag{2.24}$$

where in the second equality, the identities (2.22) have been used. The system (2.24) along with (2.19), when Z is decomposed as in (2.23), can also be written as the interesting coupled system

$$\widehat{\boldsymbol{M}}\begin{pmatrix} Z \\ \lambda \end{pmatrix}_t + \widehat{\boldsymbol{K}}(\boldsymbol{u})\begin{pmatrix} Z \\ \lambda \end{pmatrix}_x + \widehat{\boldsymbol{L}}(\boldsymbol{v})\begin{pmatrix} Z \\ \lambda \end{pmatrix}_x = \begin{pmatrix} \nabla S(Z) \\ 0 \end{pmatrix}, \tag{2.25}$$

where

$$\widehat{\boldsymbol{M}} = \begin{pmatrix} \boldsymbol{M} & \nabla P \\ -\nabla P^* & 0 \end{pmatrix}, \quad \widehat{\boldsymbol{K}}(\boldsymbol{u}) = \begin{pmatrix} \boldsymbol{K}(\boldsymbol{u}) & \nabla Q_1 \\ -\nabla Q_1{}^* & 0 \end{pmatrix}$$

and

$$\widehat{\boldsymbol{L}}(\boldsymbol{v}) = \begin{pmatrix} \boldsymbol{L}(\boldsymbol{v}) & \nabla Q_2 \\ -\nabla Q_2{}^* & 0 \end{pmatrix}.$$

The form of equations (2.24) is reminiscent of a constrained variational principle with λ_t, λ_x and λ_y as Lagrange multipliers. In fact, the form of equations (2.24) can also be obtained by imposing the mass conservation law as a constraint. Since (2.9) is the first variation of the functional $\widehat{\mathcal{F}}(Z)$ in (2.16), the constrained problem may be addressed using Lagrange multiplier theory. Therefore, one seeks stationary points of $\mathcal{F}$ subject to the constraint (2.19). Let $\lambda(x, y, t)$ represent the Lagrange multiplier associated with the constraint (2.19). Then the Lagrange necessary condition is

$$\mathrm{d}\mathcal{I}(Z; \lambda) = 0, \tag{2.26}$$

where

$$\mathcal{I}(Z;\lambda) = \int_{t_1}^{t_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} \left[\mathcal{F}(Z, Z_x, Z_y, Z_t) + \lambda(x,y,t)\left(\frac{\partial P}{\partial t} + \frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y}\right)\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}y.$$

Evaluating the first variation (2.26) results in

$$\boldsymbol{M}Z_t + \boldsymbol{K}(\boldsymbol{u})Z_x + \boldsymbol{L}(\boldsymbol{v})Z_y = \nabla S(Z) - \lambda_t \nabla P(Z) - \lambda_x \nabla Q_1(Z) - \lambda_y \nabla Q_2(Z), \tag{2.27}$$

which is precisely (2.24). The argument leading to the $\lambda = 0$ part of (2.27) follows from the least action principle associated with (2.16). However, the λ-part is not obvious and requires some explanation. First, note that

$$P_t = \langle \nabla P(Z), Z_t\rangle_m, \quad Q_{1x} = \langle \nabla Q_1(Z), Z_x\rangle_m \quad \text{and} \quad Q_{2y} = \langle \nabla Q_2(Z), Z_y\rangle_m, \tag{2.28}$$

where $\langle\cdot\,,\cdot\rangle_m$ is the inner product (2.10). Let

$$\mathcal{I}_\lambda(Z,\lambda) = \int_{t_1}^{t_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} \lambda(x,y,t)\left(\frac{\partial P}{\partial t} + \frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

Then, using (2.28), it is immediate that

$$\mathcal{I}_\lambda(Z;\lambda) = \int_{t_1}^{t_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} \lambda(x,y,t)(\langle \nabla P, Z_t\rangle_m + \langle \nabla Q_1, Z_x\rangle_m + \langle \nabla Q_2, Z_y\rangle_m)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathcal{I}_\lambda(Z+\epsilon\xi;\lambda)\bigg|_{\epsilon=0} = \int_{t_1}^{t_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} \lambda(\langle \nabla P, \xi_t\rangle_m + \langle \nabla Q_1, \xi_x\rangle_m + \langle \nabla Q_2, \xi_y\rangle_m)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$
$$+ \int_{t_1}^{t_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} \lambda(\langle D^2P\xi, Z_t\rangle_m + \langle D^2Q_1\xi, Z_x\rangle_m + \langle D^2Q_2\xi, Z_y\rangle_m)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

But, integration by parts on the second term and use of fixed endpoint conditions (variations ξ vanish on the edges of the cube: $(x_1, x_2) \times (y_1, y_2) \times (t_1, t_2)$) results in

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathcal{I}_\lambda(Z+\epsilon\xi;\lambda)\bigg|_{\epsilon=0} = -\int_{t_1}^{t_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} [\lambda_t\langle dP, \xi\rangle_m + \lambda_x\langle \nabla Q_1, \xi\rangle_m + \lambda_y\langle \nabla Q_2, \xi\rangle_m]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

and therefore the expression given in (2.27).

The Lagrange multiplier $\lambda(x, y, t)$ is related to the physical problem and governing equations as follows. Since

$$u = \phi_x + \lambda_x \quad \text{and} \quad v = \phi_y + \lambda_y,$$

it follows that (λ_x, λ_y) are components of the mean velocity and therefore

$$\phi_{xx} + \phi_{yy} + \phi_{zz} + \lambda_{xx} + \lambda_{yy} = 0.$$

Since $\Delta\phi = 0$, it follows that

$$\lambda_{xx} + \lambda_{yy} = 0. \tag{2.29}$$

However, λ_x and λ_y are representative of mean velocities and therefore should be *bounded* for all $(x, y) \in \boldsymbol{R}^2$. The only such bounded solutions of (2.29) are those where λ_x and λ_y are functions of time only. Therefore, λ takes the following form:

$$\lambda(x, y, t) = a_0(t) + a_1(t)x + a_2(t)y. \tag{2.30}$$

In the present formulation we will define the mean flow to be the flow along the group orbit function $\lambda(x, y, t)$, with the precise form (2.30), as well its dual functions, the components of the mass conservation law. In the next section the simplest non-trivial form for the mean flow is considered; taking a_0, a_1 and a_2 to be constants.

3. A variational principle for three-dimensional space and time periodic waves

In this section we treat in detail the case where the waves are periodic in space and time and the mean flow is a flow of the form

$$\lambda(x, y, t) = \gamma t + m_1 x + m_2 y, \tag{3.1}$$

where γ, m_1, m_2 are real numbers (cf. (2.30)). In particular, the multisymplectic structure and mean-flow formulation of § 2 lead naturally to a variational principle for space and time periodic states.

The form of the mean flow (3.1) is equivalent to introducing the transformation

$$\phi(x, y, z, t) = \gamma t + m_1 x + m_2 y + \tilde{\phi}(x, y, z, t) \tag{3.2}$$

in the governing equations (2.1)–(2.3). Under transformation (3.2), Laplace's equation (2.1) and the bottom boundary condition remain the same with ϕ replaced by $\tilde{\phi}$. The free surface conditions are transformed to

$$\left.\begin{aligned} &\eta_t + m_1\eta_x + m_2\eta_y + \tilde{\phi}_x\eta_x + \tilde{\phi}_y\eta_y = 0,\\ &\tilde{\phi}_t + \tfrac{1}{2}(\tilde{\phi}_x^2 + \tilde{\phi}_y^2 + \tilde{\phi}_z^2) + m_1\tilde{\phi}_x + m_2\tilde{\phi}_y + g\eta\\ &\qquad -\sigma(w_{1x} + w_{2y}) + \gamma + \tfrac{1}{2}(m_1^2 + m_2^2) = 0. \end{aligned}\right\} \tag{3.3}$$

Therefore we redefine the vector Z in (2.5) to be

$$Z = \begin{pmatrix} \Phi \\ \eta \\ w_1 \\ w_2 \\ \tilde{\phi} \\ u \\ v \end{pmatrix}, \quad \text{where} \quad \begin{cases} \Phi = \tilde{\phi}|_{z=\eta}, \\ u = \tilde{\phi}_x + m_1, \\ v = \tilde{\phi}_y + m_2, \\ \boldsymbol{u} = u|_{z=\eta}, \\ \boldsymbol{v} = v|_{z=\eta}, \end{cases} \tag{3.4}$$

in which case the governing equations (2.9) (respectively (2.24) with λ given by (3.1)) become

$$\boldsymbol{M}Z_t + \boldsymbol{K}(\boldsymbol{u})Z_x + \boldsymbol{L}(\boldsymbol{v})Z_y = \nabla S(Z) - \gamma\nabla P(Z) - \boldsymbol{m}\cdot\nabla\boldsymbol{Q}(Z) \tag{3.5}$$

where $\boldsymbol{m}\cdot\nabla\boldsymbol{Q} = m_1\nabla Q_1 + m_2\nabla Q_2$. The form (3.5) can also be used to give an interesting characterization of uniform flows. Uniform flows are solutions that are independent of (x, y, t) and, therefore, in terms of (3.5), satisfy

$$\nabla S(Z) - \gamma\nabla P(Z) - m_1\nabla Q_1(Z) - m_2\nabla Q_2(Z) = 0,$$

which can be interpreted as the Lagrange necessary condition for a constrained variational principle for uniform flows (cf. Appendix A).

Since ϕ has been decomposed into a mean flow part and a periodic part, the function $\tilde{\phi}(x, y, z, t)$ is taken to be strictly periodic in (x, y, t) of the form

$$\begin{aligned} \tilde{\phi}(x + 2\pi/k_1, y, z, t) &= \tilde{\phi}(x, y, z, t), \\ \tilde{\phi}(x, y + 2\pi/k_2, z, t) &= \tilde{\phi}(x, y, z, t), \\ \tilde{\phi}(x, y, z, t + 2\pi/\omega) &= \tilde{\phi}(x, y, z, t), \end{aligned}$$

and η is taken to be periodic of the same form. We scale x, y and t so that the frequency and wavenumbers appear in the equations as coefficients (for example, let $t' = \omega t$, $x' = k_1 x$ and $y' = k_2 y$ and then drop the primes). Then, (3.5) is transformed to

$$\omega\boldsymbol{M}Z_t + k_1\boldsymbol{K}(\boldsymbol{u})Z_x + k_2\boldsymbol{L}(\boldsymbol{v})Z_y = \nabla S(Z) - \gamma\nabla P(Z) - \boldsymbol{m}\cdot\nabla\boldsymbol{Q}(Z). \tag{3.6}$$

The vector-valued function Z is now 2π-periodic in x, y and t. Since $\boldsymbol{M}Z_t$, $\boldsymbol{K}(\boldsymbol{u})Z_x$ and $\boldsymbol{L}(\boldsymbol{v})Z_y$ are gradients of action functionals (cf. equations (2.13) and (2.15)), equation (3.6) is reminiscent of the Lagrange necessary condition for a constrained variational principle. Define the averaging operator

$$\int_{\boldsymbol{T}^3} f(x, y, t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f(x, y, t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

and introduce the constraint sets

$$\left.\begin{aligned} \int_{\boldsymbol{T}^3} A(Z,Z_t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = I_1, \quad \int_{\boldsymbol{T}^3} B_1(Z,Z_x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = I_2, \\ \int_{\boldsymbol{T}^3} B_2(Z,Z_y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = I_3, \quad \int_{\boldsymbol{T}^3} P(Z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = I_4, \\ \int_{\boldsymbol{T}^3} Q_1(Z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = I_5, \quad \int_{\boldsymbol{T}^3} Q_2(Z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = I_6, \end{aligned}\right\} \tag{3.7}$$

where $(I_1, \ldots, I_6)$ are values for the level sets of the six constraint sets.

Variational principle for three-dimensional $\boldsymbol{T}^3$-periodic waves with mean flow effects. Let $I \in \boldsymbol{R}^6$ be values for the level sets of the six functionals $(A, \boldsymbol{B}, P, \boldsymbol{Q})$, averaged over $\boldsymbol{T}^3$. Space and time periodic solutions of the specified water wave problem (with fixed values of the gravitational, depth and surface tension constants), and mean flow of the form (3.1), correspond, formally, to critical points of the functional S, averaged over $\boldsymbol{T}^3$, restricted to level sets of the functionals (3.7). Let

$$\mathcal{C}(I) = \{Z : \overline{A} = I_1, \overline{B}_1 = I_2, \overline{B}_2 = I_3, \overline{P} = I_4, \overline{Q}_1 = I_5, \overline{Q}_2 = I_6\},$$

where the overbar indicates integration over $\boldsymbol{T}^3$ as in (3.7). Then, the constrained variational formulation is

$$\mathrm{crit}\{\mathcal{S} : Z \in \mathcal{C}(I)\}, \tag{3.8}$$

where $\mathcal{S}$ is S averaged over $\boldsymbol{T}^3$. Or, introducing Lagrange multipliers $(\omega, \boldsymbol{k}, \gamma, \boldsymbol{m})$, the Lagrange necessary condition for an extremum is $\nabla\mathcal{F} = 0$, where

$$\mathcal{F}(Z; \omega, \boldsymbol{k}, \gamma, \boldsymbol{m}) = \int_{\boldsymbol{T}^3} [S - \omega A - \boldsymbol{k}\cdot\boldsymbol{B} - \gamma P - \boldsymbol{m}\cdot\boldsymbol{Q}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \tag{3.9}$$

The Lagrange necessary condition $\nabla\mathcal{F} = 0$ recovers the governing equation (3.6). It is also interesting to note that the above variational principle is similar to the variational principles for invariant tori in finite-dimensional Hamiltonian systems. The analogy between variational principles for waves, in the absence of mean-flow effects, and invariant tori is discussed in Bridges (1996*a*).

The formulation of the problem in terms of a variational principle leads to a number of interesting results, particularly with respect to the role of the parameters (the Lagrange multipliers and the values of the level sets corresponding to the constraints) in the problem. Before considering the parameter structure, the non-degeneracy condition for the constrained variational principle is introduced.

Note that a solution of $\nabla\mathcal{F} = 0$ will have the form

$$\widetilde{Z}(x, y, t; \omega, k_1, k_2, \gamma, m_1, m_2)$$

and will be 2π-periodic in (x, y, t). Values for the Lagrange multipliers are fixed by the values of the level sets corresponding to the constraints. Substitution of the above expression into (3.7) results in the equations

$$\begin{aligned} \overline{A}(\widetilde{Z}(\cdot; \omega, k_1, k_2, \gamma, m_1, m_2)) = I_1, \quad \overline{B}_1(\widetilde{Z}(\cdot; \omega, k_1, k_2, \gamma, m_1, m_2)) = I_2, \\ \overline{B}_2(\widetilde{Z}(\cdot; \omega, k_1, k_2, \gamma, m_1, m_2)) = I_3, \quad \overline{P}(\widetilde{Z}(\cdot; \omega, k_1, k_2, \gamma, m_1, m_2)) = I_4, \\ \overline{Q}_1(\widetilde{Z}(\cdot; \omega, k_1, k_2, \gamma, m_1, m_2)) = I_5, \quad \overline{Q}_2(\widetilde{Z}(\cdot; \omega, k_1, k_2, \gamma, m_1, m_2)) = I_6, \end{aligned}$$

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which are solvable for $(\omega, k_1, k_2, \gamma, m_1, m_2)$ if, and only if, (implicit function theorem)

$$\det(\boldsymbol{D}) \neq 0, \quad \text{where } \boldsymbol{D} = \begin{pmatrix} \dfrac{\partial \overline{A}}{\partial \omega} & \dfrac{\partial \overline{A}}{\partial k_1} & \dfrac{\partial \overline{A}}{\partial k_2} & \dfrac{\partial \overline{A}}{\partial \gamma} & \dfrac{\partial \overline{A}}{\partial m_1} & \dfrac{\partial \overline{A}}{\partial m_2} \\ \dfrac{\partial \overline{B_1}}{\partial \omega} & \dfrac{\partial \overline{B_1}}{\partial k_1} & \dfrac{\partial \overline{B_1}}{\partial k_2} & \dfrac{\partial \overline{B_1}}{\partial \gamma} & \dfrac{\partial \overline{B_1}}{\partial m_1} & \dfrac{\partial \overline{B_1}}{\partial m_2} \\ \dfrac{\partial \overline{B_2}}{\partial \omega} & \dfrac{\partial \overline{B_2}}{\partial k_1} & \dfrac{\partial \overline{B_2}}{\partial k_2} & \dfrac{\partial \overline{B_2}}{\partial \gamma} & \dfrac{\partial \overline{B_2}}{\partial m_1} & \dfrac{\partial \overline{B_2}}{\partial m_2} \\ \dfrac{\partial \overline{P}}{\partial \omega} & \dfrac{\partial \overline{P}}{\partial k_1} & \dfrac{\partial \overline{P}}{\partial k_2} & \dfrac{\partial \overline{P}}{\partial \gamma} & \dfrac{\partial \overline{P}}{\partial m_1} & \dfrac{\partial \overline{P}}{\partial m_2} \\ \dfrac{\partial \overline{Q_1}}{\partial \omega} & \dfrac{\partial \overline{Q_1}}{\partial k_1} & \dfrac{\partial \overline{Q_1}}{\partial k_2} & \dfrac{\partial \overline{Q_1}}{\partial \gamma} & \dfrac{\partial \overline{Q_1}}{\partial m_1} & \dfrac{\partial \overline{Q_1}}{\partial m_2} \\ \dfrac{\partial \overline{Q_2}}{\partial \omega} & \dfrac{\partial \overline{Q_2}}{\partial k_1} & \dfrac{\partial \overline{Q_2}}{\partial k_2} & \dfrac{\partial \overline{Q_2}}{\partial \gamma} & \dfrac{\partial \overline{Q_2}}{\partial m_1} & \dfrac{\partial \overline{Q_2}}{\partial m_2} \end{pmatrix}. \tag{3.10}$$

This is the non-degeneracy condition for the constrained variational principal (3.8).

The inverse of $\boldsymbol{D}$ (when defined) is the Jacobian of $(\omega, k_1, k_2, \gamma, m_1, m_2)$ with respect to $(I_1, \ldots, I_6)$, or

$$\boldsymbol{D}^{-1} = \begin{pmatrix} \dfrac{\partial \omega}{\partial I_1} & \dfrac{\partial \omega}{\partial I_2} & \dfrac{\partial \omega}{\partial I_3} & \dfrac{\partial \omega}{\partial I_4} & \dfrac{\partial \omega}{\partial I_5} & \dfrac{\partial \omega}{\partial I_6} \\ \dfrac{\partial k_1}{\partial I_1} & \dfrac{\partial k_1}{\partial I_2} & \dfrac{\partial k_1}{\partial I_3} & \dfrac{\partial k_1}{\partial I_4} & \dfrac{\partial k_1}{\partial I_5} & \dfrac{\partial k_1}{\partial I_6} \\ \dfrac{\partial k_2}{\partial I_1} & \dfrac{\partial k_2}{\partial I_2} & \dfrac{\partial k_2}{\partial I_3} & \dfrac{\partial k_2}{\partial I_4} & \dfrac{\partial k_2}{\partial I_5} & \dfrac{\partial k_2}{\partial I_6} \\ \dfrac{\partial \gamma}{\partial I_1} & \dfrac{\partial \gamma}{\partial I_2} & \dfrac{\partial \gamma}{\partial I_3} & \dfrac{\partial \gamma}{\partial I_4} & \dfrac{\partial \gamma}{\partial I_5} & \dfrac{\partial \gamma}{\partial I_6} \\ \dfrac{\partial m_1}{\partial I_1} & \dfrac{\partial m_1}{\partial I_2} & \dfrac{\partial m_1}{\partial I_3} & \dfrac{\partial m_1}{\partial I_4} & \dfrac{\partial m_1}{\partial I_5} & \dfrac{\partial m_1}{\partial I_6} \\ \dfrac{\partial m_2}{\partial I_1} & \dfrac{\partial m_2}{\partial I_2} & \dfrac{\partial m_2}{\partial I_3} & \dfrac{\partial m_2}{\partial I_4} & \dfrac{\partial m_2}{\partial I_5} & \dfrac{\partial m_2}{\partial I_6} \end{pmatrix}$$

$$= \mathrm{Hess}_I(\mathcal{S}) \stackrel{\text{def}}{=} \begin{pmatrix} \dfrac{\partial^2 \mathcal{S}}{\partial I_1^2} & \cdots & \dfrac{\partial^2 \mathcal{S}}{\partial I_1 \partial I_6} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 \mathcal{S}}{\partial I_6 \partial I_1} & \cdots & \dfrac{\partial^2 \mathcal{S}}{\partial I_6^2} \end{pmatrix}. \tag{3.11}$$

The matrix $\boldsymbol{D}$, as well as the matrix $\mathrm{Hess}_I(\mathcal{S})$, will also appear in the linear stability analysis in §4. The second equality in (3.11) is not immediately evident and will be verified shortly. It shows that, at a given space- and time-periodic wave, the

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Hamiltonian functional $\mathcal{S}$ can be characterized as an explicit function of the six parameters $(I_1, \ldots, I_6)$; the values of the level sets in equation (3.7).

The functional dependence of ω, $\boldsymbol{k}$, γ and $\boldsymbol{m}$ on $I_1, \ldots, I_6$ leads to the interesting identities

$$\omega = \frac{\partial \mathcal{S}}{\partial I_1}, \quad k_j = \frac{\partial \mathcal{S}}{\partial I_{j+1}}, \quad \gamma = \frac{\partial \mathcal{S}}{\partial I_4} \quad \text{and} \quad m_j = \frac{\partial \mathcal{S}}{\partial I_{4+j}}, \quad j = 1, 2. \tag{3.12}$$

In other words, the parameters $\omega, \boldsymbol{k}, \gamma$ and $\boldsymbol{m}$ can be characterized in terms of the slopes of curves in the parameter space $(\mathcal{S}, I_1, \ldots, I_6)$. For example, the Bernoulli constant is the rate of change of $\mathcal{S}$ with respect to the mass density with fixed actions, mass and mass flux. We will verify the first identity in (3.12) with the other three verifiable in a similar manner. Supposing that $\det(\boldsymbol{D}) \neq 0$, $\omega, \boldsymbol{k}, \gamma$ and $\boldsymbol{m}$ can be expressed as functions of $I_1, \ldots, I_6$. Back substitution into the solution vector Z results in $Z(x, y, t; I_1, \ldots, I_4)$. Evaluating the constraint sets on this solution we have

$$\left.\begin{aligned} \left\langle\!\left\langle \nabla A(Z), \frac{\partial Z}{\partial I_1} \right\rangle\!\right\rangle_m = 1, \quad \left\langle\!\left\langle \nabla \boldsymbol{B}(Z), \frac{\partial Z}{\partial I_1} \right\rangle\!\right\rangle_m = 0, \\ \left\langle\!\left\langle \nabla P(Z), \frac{\partial Z}{\partial I_1} \right\rangle\!\right\rangle_m = 0, \quad \left\langle\!\left\langle \nabla \boldsymbol{Q}(Z), \frac{\partial Z}{\partial I_1} \right\rangle\!\right\rangle_m = 0, \end{aligned}\right\} \tag{3.13}$$

where

$$\langle\!\langle \cdot\,, \cdot \rangle\!\rangle_m = \int_{\boldsymbol{T}^3} \langle \cdot\,, \cdot \rangle_m \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \tag{3.14}$$

Multiply each term in (3.13) by its corresponding Lagrange multiplier and sum to find

$$\omega = \left\langle\!\left\langle \omega \nabla A + \boldsymbol{k} \cdot \nabla \boldsymbol{B} + \gamma \nabla P + \boldsymbol{m} \cdot \nabla \boldsymbol{Q}, \frac{\partial Z}{\partial I_1} \right\rangle\!\right\rangle_m = \left\langle\!\left\langle \nabla \mathcal{S}(Z), \frac{\partial Z}{\partial I_1} \right\rangle\!\right\rangle_m = \frac{\partial \mathcal{S}}{\partial I_1},$$

proving the claim. The expressions in (3.12) also verify the second equality in (3.11).

The variational principle (3.8) formally applies to space- and time-periodic waves of arbitrary amplitude within the limitations of the formulation (for example, the free surface is required to be a single-valued function of position). The variational principle also includes standing waves and short-crested Stokes waves as well as other states that are periodic in both space and time. For the linear instability theory presented in § 3 (for uniformly travelling waves), an explicit expression for the state given by the variational principle is not necessary. The linear instability theory will be characterized in terms of the structure of the variational principle; in particular, in terms of the entries in the Jacobian matrix $\boldsymbol{D}$.

It is natural to inquire about the second derivative test for the constrained variational principle (3.8). Can the critical points given by the constrained variational principle be characterized as minima of the functional $\mathcal{S}$ on the assigned level sets? In cases where the second variation has a well-defined Morse index (finite number of negative eigenvalues) Maddocks (1996) has shown, for general constrained variational problems, that the Jacobian matrix of partial derivatives of the constraint sets with respect to the Lagrange multipliers can be used to determine critical point type. In the present problem, all critical points of the variational principle (3.8) are saddles with a countable number of positive and negative eigenvalues. In other words, the second variation of the functional $\mathcal{F}$ defines a linear operator $\mathcal{L}_0 = D^2\mathcal{F}(Z)$ which has a countable number of negative and positive eigenvalues. (This can be easily verified by considering the linear operator $D^2\mathcal{F}(0)$, where the spectrum can be explicitly

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computed. This difficulty is similar to the difficulty associated with the use of Hamilton's principle for the existence of periodic orbits of finite-dimensional Hamiltonian systems.) However, although the matrix $\boldsymbol{D}$ contains no information about critical point type, it does contain information about the linear stability exponents for the extremals. To prove this we have to work directly with the space- and time-dependent governing equations (3.5) perturbed about the basic state.

4. Instability of three-dimensional travelling waves with mean flow effects and arbitrary amplitude

In this section the linear instability problem for periodic travelling waves is formulated in terms of the multisymplectic structure and Hamiltonian formalism introduced in § 2 and an explicit relation is established between the linear stability exponents and the variational principle presented in § 3.

Suppose there exists a periodic travelling wave obtained by the variational principle (3.8) with governing equation (3.6). The basic state can be written as

$$\{\widehat{Z}(\theta; I),\ \omega(I),\ \boldsymbol{k}(I),\ \gamma(I),\ \boldsymbol{m}(I) : I = (I_1, \ldots, I_6) \text{ and } \theta = x + y - t\}. \tag{4.1}$$

To study the instability of the state (4.1), let $Z = \widehat{Z} + \widehat{U}$, substitute into the basic equation (3.5) and linearize about $\widehat{Z}$ to obtain

$$\begin{aligned}\boldsymbol{M}\widehat{U}_{\tilde{t}} + \boldsymbol{K}(\widehat{\boldsymbol{u}})\widetilde{U}_{\tilde{x}} + \{D\boldsymbol{K}(\widehat{\boldsymbol{u}}), \widehat{U}\}\hat{Z}_{\tilde{x}} + \boldsymbol{L}(\widehat{\boldsymbol{v}})\widetilde{U}_{\tilde{y}} + \{D\boldsymbol{L}(\widehat{\boldsymbol{v}}), \widehat{U}\}\hat{Z}_{\tilde{y}} \\ = [D^2 S(\hat{Z}) - \gamma D^2 \widehat{P}(\hat{Z}) - \boldsymbol{m}\cdot D^2\widehat{\boldsymbol{Q}}(\hat{Z})]\widehat{U}.\end{aligned} \tag{4.2}$$

Note that the above equation is not periodic in $\tilde{x}$, $\tilde{y}$ and $\tilde{t}$, and this fact is indicated by the tilde over x, y and t. Equation (4.2) is, however, a linear partial differential equation with periodic coefficients and, therefore, we will invoke the Floquet ansatz and consider solutions of (4.2) of the form

$$\widehat{U} = \mathrm{Re}(U\mathrm{e}^{\mathrm{i}(p_1\tilde{x} + p_2\tilde{y} - \Omega\tilde{t})}), \tag{4.3}$$

where U is a periodic function of $\tilde{x}$, $\tilde{y}$, $\tilde{t}$, $\boldsymbol{p} \in \boldsymbol{R}^2$ (perturbations should be bounded for all $(\tilde{x}, \tilde{y}) \in \boldsymbol{R}^2$) and Ω is, in general, complex. It should be noted that although the Floquet ansatz is convenient it is not essential. The Floquet theorem, when applicable, states that all bounded solutions of the PDE (4.2) have a decomposition (4.3), whereas, in the present theory, being an instability theory, it is only necessary to show the existence of a single solution of the form (4.3) with an unstable stability exponent.

To coincide with the analysis in § 2, the variables $\tilde{x}$, $\tilde{y}$ and $\tilde{t}$ are now scaled so that ω and $\boldsymbol{k}$ appear in the equations and U is then a 2π-periodic function in x, y and t. Define the linear operator $\mathcal{L}_0$ by

$$\mathcal{L}_0 = D^2 S(\hat{Z}) - \omega D^2 A(\hat{Z}) - \boldsymbol{k}\cdot D^2\boldsymbol{B}(\hat{Z}) - \gamma D^2 P(\hat{Z}) - \boldsymbol{m}\cdot D^2\boldsymbol{Q}(\hat{Z}). \tag{4.4}$$

Then, substitution of (4.3) into (4.2) with $x = k_1\tilde{x}$, $y = k_2\tilde{y}$ and $t = \omega\tilde{t}$ results in

$$\mathcal{L}_0 U = -\mathrm{i}\Omega\boldsymbol{M}U + \mathrm{i}p_1\boldsymbol{K}(\widehat{\boldsymbol{u}})U + \mathrm{i}p_2\boldsymbol{L}(\widehat{\boldsymbol{v}})U. \tag{4.5}$$

The linear instability problem for the basic state in (4.1) can be stated as follows. Fixing the basic state (of the form (4.1)) in equation (4.5), we say that the basic state is linearly unstable if for some $\boldsymbol{p} \in \boldsymbol{R}^2$ there exists a periodic function $U(x, y, t)$ satisfying (4.5) with $\mathrm{Im}(\Omega) \neq 0$. The present theory is based on the fact that, for

$|\Omega|$ and $|\boldsymbol{p}|$ sufficiently small, the problem (4.5) can be completely analysed. In this section we will establish the following result.

Linear instability criterion. Let $(\widehat{Z}; \omega, \boldsymbol{k}, \gamma, \boldsymbol{m})$ be a basic state of the form (4.1) with corresponding Jacobian matrix $\boldsymbol{D}$ defined in (3.10) partitioned according to the matrices $\boldsymbol{N}_{2j}$, $j = 1, 2, 3$ in (1.11). Let

$$\Upsilon(\Omega, \boldsymbol{p}) = \Upsilon_2(\Omega, \boldsymbol{p}) + R_5(\Omega, \boldsymbol{p}),$$

with $R_5(\Omega, \boldsymbol{p})$ as defined in (1.9) and

$$\Upsilon_2(\Omega, \boldsymbol{p}) = \det[\boldsymbol{N}_2(\Omega, \boldsymbol{p})],$$

with $\boldsymbol{N}_2(\Omega, \boldsymbol{p})$ as defined in (1.10). If for $\boldsymbol{p} \in \boldsymbol{R}^2$ and $|\boldsymbol{p}|$ sufficiently small there exists a solution of $\Upsilon_2(\Omega, \boldsymbol{p}) = 0$ with $\mathrm{Im}(\Omega) \neq 0$, then the basic state (4.1) is linearly unstable.

The linear instability criterion is verified by constructing an unstable eigenfunction using the basic state and derivatives of the basic state with respect to θ and the parameters $I_1, \ldots, I_6$. Note first that when $\Omega = \boldsymbol{p} = 0$, equation (4.5) has two independent periodic solutions:

$$\mathcal{L}_0 \Psi_0 = 0 \quad \text{and} \quad \mathcal{L}_0 V = 0, \tag{4.6}$$

where

$$\Psi_0 = \frac{\partial \hat{Z}}{\partial \theta} \quad \text{and} \quad V = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.7}$$

The function Ψ_0 is the tangent vector to the basic state and V is the tangent vector to the group orbit associated with constant perturbation of the potential (cf. equation (2.17)). This seemingly trivial symmetry is fundamental to the analysis. It is crucially related to the mass conservation law and generates the three functionals P and $\boldsymbol{Q}$ (see equations (2.19)–(2.22) and discussion there). We take as a hypothesis that $\{\Psi_0, V\}$ are the only periodic elements in the kernel of $\mathcal{L}_0$.

The strategy of the analysis is as follows. When $\Omega = \boldsymbol{p} = 0$ in (4.5) the kernel of $\mathcal{L}_0$ restricted to periodic functions is spanned by the two functions Ψ_0 and V. Formally, the operator $\mathcal{L}_0$ is self-adjoint, being the second variation of the functional $\mathcal{F}$, and therefore equation (4.5) is solvable if, and only if, the inner product of the right-hand side with Ψ_0 and V vanishes. Restriction of $\mathcal{L}_0$ to the complement of the kernel of $\mathcal{L}_0$ is formally solvable as a power series in Ω and $\boldsymbol{p}$. Back substitution of the complementary function into the solvability conditions results in a dispersion relation for the linear stability problem, $\Upsilon(\Omega, \boldsymbol{p}) = 0$. With a precise specification of the function spaces on which (4.5) is well defined, the above procedure would be the Lyapunov–Schmidt method. For example, an analysis of this type has been used in Bridges & Rowlands (1994) to give a rigorous instability analysis of spatially quasiperiodic states of the Ginzburg–Landau equation, and in Bridges (1996*b*, § 4) to give a rigorous proof for general nonlinear dispersive wave problems when mean flow effects are absent. However, although it seems tractable to identify the function spaces on which (4.5) is well defined, we do not consider this problem here and will proceed formally with the analysis.

Write the general solution of (4.5) in the form

$$U = \alpha_1 \Psi_0 + \alpha_2 V + W, \quad \text{with} \quad W|_{\Omega=\boldsymbol{p}=0} = 0, \tag{4.8 a}$$

where α_1 and α_2 are, at this point, arbitrary complex numbers. Since the problem (4.5) is linear, W will depend linearly on α_1 and α_2 and therefore

$$W = \alpha_1 W_1 + \alpha_2 W_2. \tag{4.8 b}$$

Formally, equation (4.5) is solvable if, and only if, the inner product of the right-hand side of (4.5) with Ψ_0 and V vanishes, resulting in the two conditions

$$\left.\begin{array}{r}[\Psi_0, -\mathrm{i}\Omega \boldsymbol{M} U + \mathrm{i}p_1 \boldsymbol{K}(\widehat{\boldsymbol{u}})U + \mathrm{i}p_2 \boldsymbol{L}(\widehat{\boldsymbol{v}})U] = 0,\\ [V, -\mathrm{i}\Omega \boldsymbol{M} U + \mathrm{i}p_1 \boldsymbol{K}(\widehat{\boldsymbol{u}})U + \mathrm{i}p_2 \boldsymbol{L}(\widehat{\boldsymbol{v}})U] = 0,\end{array}\right\} \tag{4.9}$$

with U given in (4.8). The inner product in this case is taken to be

$$[\cdot\,,\cdot] \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} \langle\cdot\,,\cdot\rangle_m \,\mathrm{d}\theta.$$

Substitution of (4.8) into (4.9) results in the algebraic equation

$$[\boldsymbol{N}(\Omega, p)] \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0, \tag{4.10}$$

where $\boldsymbol{N}(\Omega, \boldsymbol{p}) = 0$ is a 2×2 matrix with, in general, complex-valued entries. The algebraic equation (4.10) has a non-trivial solution if, and only if,

$$\Upsilon(\Omega, \boldsymbol{p}) \stackrel{\text{def}}{=} \det[\boldsymbol{N}(\Omega, \boldsymbol{p})] = 0. \tag{4.11}$$

The condition $\Upsilon(\Omega, \boldsymbol{p}) = 0$ is the dispersion relation for the linear stability problem.

We begin the analysis of the dispersion relation by proving the following:

$$\boldsymbol{N}(\Omega, \boldsymbol{p}) = \boldsymbol{N}_2(\Omega, \boldsymbol{p}) + o(|\Omega|^2 + |\boldsymbol{p}|^2),$$

where $\boldsymbol{N}_2(\Omega, \boldsymbol{p})$ is quadratic in Ω and $\boldsymbol{p}$, i.e.

$$\boldsymbol{N}(\Omega, \boldsymbol{p})\Big|_{\Omega=\boldsymbol{p}=0} = \frac{\partial}{\partial \Omega}\boldsymbol{N}(\Omega, \boldsymbol{p})\Big|_{\Omega=\boldsymbol{p}=0} = \frac{\partial}{\partial \boldsymbol{p}}\boldsymbol{N}(\Omega, \boldsymbol{p})\Big|_{\Omega=\boldsymbol{p}=0} = 0. \tag{4.12}$$

It is evident that $\boldsymbol{N}(0,0) = 0$. To verify that the first derivatives also vanish we note that

$$\frac{\partial}{\partial \Omega}\boldsymbol{N}(\Omega, \boldsymbol{p})\Big|_{\Omega=\boldsymbol{p}=0} = -\mathrm{i}\begin{pmatrix} [\Psi_0, \boldsymbol{M}\Psi_0] & [\Psi_0, \boldsymbol{M}V] \\ [V, \boldsymbol{M}\Psi_0] & [V, \boldsymbol{M}V] \end{pmatrix},$$

$$\frac{\partial}{\partial p_1}\boldsymbol{N}(\Omega, \boldsymbol{p})\Big|_{\Omega=\boldsymbol{p}=0} = \mathrm{i}\begin{pmatrix} [\Psi_0, \boldsymbol{K}(\widehat{\boldsymbol{u}})\Psi_0] & [\Psi_0, \boldsymbol{K}(\widehat{\boldsymbol{u}})V] \\ [V, \boldsymbol{K}(\widehat{\boldsymbol{u}})\Psi_0] & [V, \boldsymbol{K}(\widehat{\boldsymbol{u}})V] \end{pmatrix},$$

$$\frac{\partial}{\partial p_2}\boldsymbol{N}(\Omega, \boldsymbol{p})\Big|_{\Omega=\boldsymbol{p}=0} = \mathrm{i}\begin{pmatrix} [\Psi_0, \boldsymbol{L}(\widehat{\boldsymbol{v}})\Psi_0] & [\Psi_0, \boldsymbol{L}(\widehat{\boldsymbol{v}})V] \\ [V, \boldsymbol{L}(\widehat{\boldsymbol{v}})\Psi_0] & [V, \boldsymbol{L}(\widehat{\boldsymbol{v}})V] \end{pmatrix}.$$

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The diagonal terms in the above expression have the form

$$[\Psi_0, \boldsymbol{M}\,\Psi_0] = [V, \boldsymbol{M}V] = 0,$$
$$[\Psi_0, \boldsymbol{K}(\widehat{\boldsymbol{u}})\,\Psi_0] = [V, \boldsymbol{K}(\widehat{\boldsymbol{u}})V] = 0,$$
$$[\Psi_0, \boldsymbol{L}(\widehat{\boldsymbol{v}})\,\Psi_0] = [V, \boldsymbol{L}(\widehat{\boldsymbol{v}})V] = 0,$$

and vanish identically due to the skew-symmetry of $\boldsymbol{M}$, $\boldsymbol{K}(\widehat{\boldsymbol{u}})$ and $\boldsymbol{L}(\widehat{\boldsymbol{v}})$. For the off-diagonal terms we have the following identities:

$$[\Psi_0, \boldsymbol{M}V] = [\Psi_0, \nabla P(\widehat{Z})] = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial P}{\partial \theta}\mathrm{d}\theta = 0,$$
$$[\Psi_0, \boldsymbol{K}(\widehat{\boldsymbol{u}})V] = [\Psi_0, \nabla Q_1(\widehat{Z})] = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial Q_1}{\partial \theta}\mathrm{d}\theta = 0,$$
$$[\Psi_0, \boldsymbol{L}(\widehat{\boldsymbol{v}})V] = [\Psi_0, \nabla Q_2(\widehat{Z})] = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial Q_2}{\partial \theta}\mathrm{d}\theta = 0,$$

where the last equality follows from the periodicity of the functionals P, Q_1 and Q_2 evaluated on the basic state. This completes the verification of (4.12).

The dispersion relation has been reduced to the following form:

$$\Upsilon(\Omega, \boldsymbol{p}) = \Upsilon_2(\Omega, \boldsymbol{p}) + R_5(\Omega, \boldsymbol{p}), \qquad (4.13\,a)$$

with

$$\Upsilon_2(\Omega, \boldsymbol{p}) = \det[\boldsymbol{N}_2(\Omega, \boldsymbol{p})]. \qquad (4.13\,b)$$

To determine the coefficients in the matrix $\boldsymbol{N}_2(\Omega, \boldsymbol{p})$ it is necessary to evaluate W to first order in Ω and $\boldsymbol{p}$. Therefore, let

$$\left.\begin{aligned} W_1 &= \mathrm{i}\Omega W_{11} + \mathrm{i}p_1 W_{12} + \mathrm{i}p_2 W_{13} + \mathcal{O}(|\Omega|^2 + |\boldsymbol{p}|^2),\\ W_2 &= -\mathrm{i}\Omega W_{21} + \mathrm{i}p_1 W_{22} + \mathrm{i}p_2 W_{23} + \mathcal{O}(|\Omega|^2 + |\boldsymbol{p}|^2). \end{aligned}\right\} \qquad (4.14)$$

Substitution into (4.5) leads to the following inhomogeneous equations for W_{ij} ($i = 1,2$; $j = 1,2,3$):

$$\left.\begin{aligned} \mathcal{L}_0 W_{11} &= -\boldsymbol{M}\,\Psi_0 = \nabla A(\widehat{Z}),\\ \mathcal{L}_0 W_{12} &= \boldsymbol{K}(\widehat{\boldsymbol{u}})\,\Psi_0 = \nabla B_1(\widehat{Z}),\\ \mathcal{L}_0 W_{13} &= \boldsymbol{L}(\widehat{\boldsymbol{v}})\,\Psi_0 = \nabla B_2(\widehat{Z}), \end{aligned}\right\} \qquad (4.15)$$

$$\left.\begin{aligned} \mathcal{L}_0 W_{21} &= -\boldsymbol{M}V = -\nabla P(\widehat{Z}),\\ \mathcal{L}_0 W_{22} &= \boldsymbol{K}(\widehat{\boldsymbol{u}})V = \nabla Q_1(\widehat{Z}),\\ \mathcal{L}_0 W_{23} &= \boldsymbol{L}(\widehat{\boldsymbol{v}})V = \nabla Q_2(\widehat{Z}). \end{aligned}\right\} \qquad (4.16)$$

The second equality in (4.15) follows from (2.15) and the second equality in (4.16) follows from (2.22).

An important part of the theoretical construction is that the inhomogeneous equations (4.15), (4.16) can be solved exactly. Each of the functions W_{ij} ($i = 1,2$; $j = 1,2,3$) can be expressed as a linear combination of the functions $\partial\widehat{Z}/\partial I_j$ for $j = 1,\ldots,6$.

Let $\Psi_j = \partial\widehat{Z}/\partial I_j$ for $j = 1,\ldots,6$. Then differentiation of (3.6) with respect to I_j

for $j=1,\ldots,6$, using the fact that $\omega, \boldsymbol{k}, \gamma$ and $\boldsymbol{m}$ depend on I, results in

$$\mathcal{L}_0 \Psi_j = \frac{\partial\omega}{\partial I_j}\nabla A(\hat{Z}) + \frac{\partial \boldsymbol{k}}{\partial I_j}\cdot\nabla \boldsymbol{B}(\hat{Z}) + \frac{\partial\gamma}{\partial I_j}\nabla P(\hat{Z}) + \frac{\partial \boldsymbol{m}}{\partial I_j}\cdot\nabla \boldsymbol{Q}(\hat{Z}), \quad j=1,\ldots,6. \quad (4.17)$$

Using the functions $\{\Psi_1,\ldots,\Psi_6\}$, the complementary solutions of equations (4.15), (4.16) are

$$\left.\begin{aligned} \alpha_1 W_{11} + \alpha_2 W_{21} &= \textstyle\sum_{j=1}^6 a_j^{(1)}\Psi_j, \\ \alpha_1 W_{12} + \alpha_2 W_{22} &= \textstyle\sum_{j=1}^6 a_j^{(2)}\Psi_j, \\ \alpha_1 W_{13} + \alpha_2 W_{23} &= \textstyle\sum_{j=1}^6 a_j^{(3)}\Psi_j. \end{aligned}\right\} \quad (4.18)$$

Substituting the first expression of (4.18) into (4.15), (4.16) leads to

$$\mathcal{L}_0(\alpha_1 W_{11} + \alpha_2 W_{21}) = \sum_{j=1}^{6} a_j^{(1)} \mathcal{L}_0 \Psi_j = \alpha_1 \nabla A(\widehat{Z}) + \alpha_2 \nabla P(\widehat{Z}),$$

or

$$\sum_{j=1}^{6} a_j^{(1)} \frac{\partial\omega}{\partial I_j}\nabla A(\widehat{Z}) + \sum_{j=1}^{6} a_j^{(1)} \frac{\partial \boldsymbol{k}}{\partial I_j}\nabla \boldsymbol{B}(\widehat{Z}) + \sum_{j=1}^{6} a_j^{(1)} \frac{\partial\gamma}{\partial I_j}\nabla P(\widehat{Z}) + \sum_{j=1}^{6} a_j^{(1)} \frac{\partial \boldsymbol{m}}{\partial I_j}\nabla \boldsymbol{Q}(\widehat{Z}).$$
$$= \alpha_1 \nabla A(\widehat{Z}) + \alpha_2 \nabla P(\widehat{Z}).$$

Equating coefficients of each gradient results in

$$\begin{pmatrix} \frac{\partial\omega}{\partial I_1} & \frac{\partial\omega}{\partial I_2} & \frac{\partial\omega}{\partial I_3} & \frac{\partial\omega}{\partial I_4} & \frac{\partial\omega}{\partial I_5} & \frac{\partial\omega}{\partial I_6} \\ \frac{\partial k_1}{\partial I_1} & \frac{\partial k_1}{\partial I_2} & \frac{\partial k_1}{\partial I_3} & \frac{\partial k_1}{\partial I_4} & \frac{\partial k_1}{\partial I_5} & \frac{\partial k_1}{\partial I_6} \\ \frac{\partial k_2}{\partial I_1} & \frac{\partial k_2}{\partial I_2} & \frac{\partial k_2}{\partial I_3} & \frac{\partial k_2}{\partial I_4} & \frac{\partial k_2}{\partial I_5} & \frac{\partial k_2}{\partial I_6} \\ \frac{\partial\gamma}{\partial I_1} & \frac{\partial\gamma}{\partial I_2} & \frac{\partial\gamma}{\partial I_3} & \frac{\partial\gamma}{\partial I_4} & \frac{\partial\gamma}{\partial I_5} & \frac{\partial\gamma}{\partial I_6} \\ \frac{\partial m_1}{\partial I_1} & \frac{\partial m_1}{\partial I_2} & \frac{\partial m_1}{\partial I_3} & \frac{\partial m_1}{\partial I_4} & \frac{\partial m_1}{\partial I_5} & \frac{\partial m_1}{\partial I_6} \\ \frac{\partial m_2}{\partial I_1} & \frac{\partial m_2}{\partial I_2} & \frac{\partial m_2}{\partial I_3} & \frac{\partial m_2}{\partial I_4} & \frac{\partial m_2}{\partial I_5} & \frac{\partial m_2}{\partial I_6} \end{pmatrix} \begin{pmatrix} a_1^{(1)} \\ a_2^{(1)} \\ a_3^{(1)} \\ a_4^{(1)} \\ a_5^{(1)} \\ a_6^{(1)} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ 0 \\ 0 \\ \alpha_2 \\ 0 \\ 0 \end{pmatrix}.$$

Noting that the matrix on the left-hand side is $\boldsymbol{D}^{-1} = \mathrm{Hess}_I(S)$, it follows that

$$\boldsymbol{a}^{(1)} \stackrel{\mathrm{def}}{=} \begin{pmatrix} a_1^{(1)} \\ \vdots \\ a_6^{(1)} \end{pmatrix} = \alpha_1 \boldsymbol{D} e_1 + \alpha_2 \boldsymbol{D} e_4,$$

where the e_j are the usual unit vectors on $\boldsymbol{R}^6$. Similarly,

$$\boldsymbol{a}^{(2)} = \alpha_1 \boldsymbol{D} e_2 + \alpha_2 \boldsymbol{D} e_5, \quad \boldsymbol{a}^{(3)} = \alpha_1 \boldsymbol{D} e_3 + \alpha_2 \boldsymbol{D} e_6.$$

Note that the equations for $\boldsymbol{a}^{(j)}$, $j=1,2,3$ are solvable if, and only if, $\det(\boldsymbol{D}) \neq 0$; i.e.

provided the constrained variational principle (3.8) is non-degenerate. It is interesting that the non-degeneracy condition from the *constrained* variational principle arises in a natural way in the linear stability analysis.

Using the above expressions for $\boldsymbol{a}^{(j)}$, $j = 1, 2, 3$, the expressions in (4.18) can be recast into the form

$$W_{11} = \sum_{j=1}^{6} \frac{\partial \omega}{\partial I_j} \Psi_j, \quad W_{12} = \sum_{j=1}^{6} \frac{\partial k_1}{\partial I_j} \Psi_j, \quad W_{13} = \sum_{j=1}^{6} \frac{\partial k_2}{\partial I_j} \Psi_j,$$

$$W_{21} = \sum_{j=1}^{6} \frac{\partial \gamma}{\partial I_j} \Psi_j, \quad W_{22} = \sum_{j=1}^{6} \frac{\partial m_1}{\partial I_j} \Psi_j, \quad W_{23} = \sum_{j=1}^{6} \frac{\partial m_2}{\partial I_j} \Psi_j,$$

and therefore

$$\left.\begin{aligned} W_1 &= \sum_{j=1}^{6} \left(\mathrm{i}\Omega \frac{\partial \omega}{\partial I_j} + \mathrm{i}\boldsymbol{p} \cdot \frac{\partial \boldsymbol{k}}{\partial I_j} \right) \Psi_j + \mathcal{O}(|\Omega|^2 + |\boldsymbol{p}|^2), \\ W_2 &= \sum_{j=1}^{6} \left(-\mathrm{i}\Omega \frac{\partial \gamma}{\partial I_j} + \mathrm{i}\boldsymbol{p} \cdot \frac{\partial \boldsymbol{m}}{\partial I_j} \right) \Psi_j + \mathcal{O}(|\Omega|^2 + |\boldsymbol{p}|^2). \end{aligned}\right\} \tag{4.19}$$

This completes the construction of the part of W that is linear in Ω and $\boldsymbol{p}$ (cf. equation (4.14)).

We now have enough information to evaluate the entries in the matrix $\boldsymbol{N}_2(\Omega, \boldsymbol{p})$, which has the form

$$\boldsymbol{N}_2(\Omega, \boldsymbol{p}) = \begin{pmatrix} [\Psi_0, \boldsymbol{J}(\Omega, \boldsymbol{p}) W_1] & [\Psi_0, \boldsymbol{J}(\Omega, \boldsymbol{p}) W_2] \\ [V, \boldsymbol{J}(\Omega, \boldsymbol{p}) W_1] & [V, \boldsymbol{J}(\Omega, \boldsymbol{p}) W_2] \end{pmatrix}, \tag{4.20}$$

where $\boldsymbol{J}(\Omega, \boldsymbol{p}) = -\mathrm{i}\Omega \boldsymbol{M} + \mathrm{i}p_1 \boldsymbol{K}(\widehat{\boldsymbol{u}}) + \mathrm{i}p_2 \boldsymbol{L}(\widehat{\boldsymbol{v}})$ and W_1 and W_2 are given by their leading order, linear in Ω and $\boldsymbol{p}$, expressions in (4.19).

In order to evaluate the terms in (4.20), it is necessary to evaluate inner products between Ψ_0, V and the functions Ψ_j, $j = 1, \ldots, 6$. It is a consequence of the variational principle of § 3 that such inner products can be evaluated exactly. We will need the following identities:

$$\left.\begin{aligned} [\Psi_0, \boldsymbol{M} \Psi_j] &= \delta_{1j}, \quad [\Psi_0, \boldsymbol{K}(\widehat{\boldsymbol{u}}) \Psi_j] = -\delta_{2j}, \quad [\Psi_0, \boldsymbol{L}(\widehat{\boldsymbol{v}}) \Psi_j] = -\delta_{3j}, \\ [V, \boldsymbol{M} \Psi_j] &= -\delta_{4j}, \quad [V, \boldsymbol{K}(\widehat{\boldsymbol{u}}) \Psi_j] = -\delta_{5j}, \quad [V, \boldsymbol{L}(\widehat{\boldsymbol{v}}) \Psi_j] = -\delta_{6j}, \end{aligned}\right\} \tag{4.21}$$

for $j = 1, \ldots, 6$, where $\delta_{ij} = 1$ if $i = j$ and zero otherwise. The verification of the first identity in (4.21) is obtained by noting that $I_1 = \overline{A}(\hat{Z})$ and

$$\delta_{1j} = \frac{\partial I_1}{\partial I_j} = \left[\nabla A(\hat{Z}), \frac{\partial \hat{Z}}{\partial I_j} \right] = \left[\boldsymbol{M} \hat{Z}_t, \frac{\partial \hat{Z}}{\partial I_j} \right] = -[\boldsymbol{M} \hat{Z}_\theta, \Psi_j] = [\Psi_0, \boldsymbol{M} \Psi_j]$$

for $j = 1, \ldots, 6$ using the definition of $\nabla A(\hat{Z})$ and the skew-symmetry of $\boldsymbol{M}$. For the fourth identity in (4.21), note that $I_4 = \overline{P}(\hat{Z})$ and

$$\delta_{4j} = \frac{\partial I_4}{\partial I_j} = \left[\nabla P(\hat{Z}), \frac{\partial \hat{Z}}{\partial I_j} \right] = \left[\boldsymbol{M} V, \frac{\partial \hat{Z}}{\partial I_j} \right] = -[V, \boldsymbol{M} \Psi_j]$$

using the relation $\nabla P(\hat{Z}) = \boldsymbol{M}V$ derived in § 2. The second, third, fifth and sixth identities in (4.21) are verified in a similar manner.

The symmetric matrix $\boldsymbol{D}$ can be partitioned into 3×3 submatrices as follows:

$$\boldsymbol{D} = \begin{pmatrix} \boldsymbol{D}_1 & \boldsymbol{D}_2 \\ \boldsymbol{D}_2^{\mathrm{T}} & \boldsymbol{D}_3 \end{pmatrix}. \tag{4.22}$$

The entries of $\boldsymbol{N}_2(\Omega, \boldsymbol{p})$ can now be evaluated as follows. Let $W_1^{(1)}$ and $W_2^{(2)}$ denote the part of W_1 and W_2 in (4.19) that is linear in Ω and $\boldsymbol{p}$. Then,

$$\begin{aligned}
[\boldsymbol{N}_2(\Omega,\boldsymbol{p})]_{(1,1)} &= [\Psi_0,\ (-i\Omega\boldsymbol{M} + \mathrm{i}p_1\boldsymbol{K}(\widehat{\boldsymbol{u}}) + \mathrm{i}p_2\boldsymbol{L}(\widehat{\boldsymbol{v}}))W_1^{(1)}] \\
&= \left[\mathrm{i}\Omega\boldsymbol{M}\,\Psi_0 - \mathrm{i}p_1\boldsymbol{K}(\widehat{\boldsymbol{u}})\,\Psi_0 - \mathrm{i}p_2\boldsymbol{L}(\widehat{\boldsymbol{v}})\,\Psi_0,\ \sum_{j=1}^{6}\left(\mathrm{i}\Omega\frac{\partial\omega}{\partial I_j} + \mathrm{i}\boldsymbol{p}\cdot\frac{\partial\boldsymbol{k}}{\partial I_j}\right)\Psi_j\right] \\
&= \sum_{j=1}^{6}\left(\mathrm{i}\Omega\frac{\partial\omega}{\partial I_j} + \mathrm{i}\boldsymbol{p}\cdot\frac{\partial\boldsymbol{k}}{\partial I_j}\right)[\mathrm{i}\Omega\boldsymbol{M}\,\Psi_0 - \mathrm{i}p_1\boldsymbol{K}(\widehat{\boldsymbol{u}})\,\Psi_0 - \mathrm{i}p_2\boldsymbol{L}(\widehat{\boldsymbol{v}})\,\Psi_0,\ \Psi_j] \\
&= \sum_{j=1}^{6}\left(\mathrm{i}\Omega\frac{\partial\omega}{\partial I_j} + \mathrm{i}\boldsymbol{p}\cdot\frac{\partial\boldsymbol{k}}{\partial I_j}\right)(-\mathrm{i}\Omega\delta_{1j} - \mathrm{i}p_1\delta_{2j} - \mathrm{i}p_2\delta_{3j}) \\
&= \left\langle \begin{pmatrix} \Omega \\ p_1 \\ p_2 \end{pmatrix}, \boldsymbol{D}_1 \begin{pmatrix} \Omega \\ p_1 \\ p_2 \end{pmatrix} \right\rangle_{\boldsymbol{R}^3}.
\end{aligned}$$

Note that the elements in the inner product $\langle\cdot,\cdot\rangle_{\boldsymbol{R}^3}$ above are complex valued. The subscript $\boldsymbol{R}^3$ indicates that no complex conjugation takes place during the product. A similar argument results in

$$\begin{aligned}
[\boldsymbol{N}_2(\Omega,\boldsymbol{p})]_{(1,1)} &= [\Psi_0,\ (-\mathrm{i}\Omega\boldsymbol{M} + \mathrm{i}p_1\boldsymbol{K}(\widehat{\boldsymbol{u}}) + \mathrm{i}p_2\boldsymbol{L}(\widehat{\boldsymbol{v}}))W_1^{(1)}] \\
&= \left[\mathrm{i}\Omega\boldsymbol{M}\,\Psi_0 - \mathrm{i}p_1\boldsymbol{K}(\widehat{\boldsymbol{u}})\,\Psi_0 - \mathrm{i}p_2\boldsymbol{L}(\widehat{\boldsymbol{v}})\,\Psi_0,\ \sum_{j=1}^{6}\left(\mathrm{i}\Omega\frac{\partial\omega}{\partial I_j} + \mathrm{i}\boldsymbol{p}\cdot\frac{\partial\boldsymbol{k}}{\partial I_j}\right)\Psi_j\right] \\
&= \sum_{j=1}^{6}\left(\mathrm{i}\Omega\frac{\partial\omega}{\partial I_j} + \mathrm{i}\boldsymbol{p}\cdot\frac{\partial\boldsymbol{k}}{\partial I_j}\right)[\mathrm{i}\Omega\boldsymbol{M}\,\Psi_0 - \mathrm{i}p_1\boldsymbol{K}(\widehat{\boldsymbol{u}})\,\Psi_0 - \mathrm{i}p_2\boldsymbol{L}(\widehat{\boldsymbol{v}})\,\Psi_0,\ \Psi_j] \\
&= \sum_{j=1}^{6}\left(\mathrm{i}\Omega\frac{\partial\omega}{\partial I_j} + \mathrm{i}\boldsymbol{p}\cdot\frac{\partial\boldsymbol{k}}{\partial I_j}\right)(-\mathrm{i}\Omega\delta_{1j} - \mathrm{i}p_1\delta_{2j} - \mathrm{i}p_2\delta_{3j}) \\
&= \left\langle \begin{pmatrix} \Omega \\ p_1 \\ p_2 \end{pmatrix}, \boldsymbol{D}_1 \begin{pmatrix} \Omega \\ p_1 \\ p_2 \end{pmatrix} \right\rangle_{\boldsymbol{R}^3}.
\end{aligned}$$

and

$$[\boldsymbol{N}_2(\Omega,\boldsymbol{p})]_{(2,2)} = \left\langle \begin{pmatrix} -\Omega \\ p_1 \\ p_2 \end{pmatrix}, \boldsymbol{D}_3 \begin{pmatrix} -\Omega \\ p_1 \\ p_2 \end{pmatrix} \right\rangle_{\boldsymbol{R}^3}.$$

Letting

$$\left.\begin{aligned} d_1(\Omega,\boldsymbol{p}) &= \left\langle \begin{pmatrix} \Omega \\ \boldsymbol{p} \end{pmatrix}, \boldsymbol{D}_1 \begin{pmatrix} \Omega \\ \boldsymbol{p} \end{pmatrix} \right\rangle_{\boldsymbol{R}^3}, \\ d_2(\Omega,\boldsymbol{p}) &= \left\langle \begin{pmatrix} \Omega \\ \boldsymbol{p} \end{pmatrix}, \boldsymbol{D}_2 \begin{pmatrix} -\Omega \\ \boldsymbol{p} \end{pmatrix} \right\rangle_{\boldsymbol{R}^3}, \\ d_3(\Omega,\boldsymbol{p}) &= \left\langle \begin{pmatrix} -\Omega \\ \boldsymbol{p} \end{pmatrix}, \boldsymbol{D}_3 \begin{pmatrix} -\Omega \\ \boldsymbol{p} \end{pmatrix} \right\rangle_{\boldsymbol{R}^3}, \end{aligned}\right\} \tag{4.23}$$

the resulting expression for $\boldsymbol{N}_2(\Omega,\boldsymbol{p})$ is then

$$\begin{aligned} \boldsymbol{N}_2(\Omega,\boldsymbol{p}) &= \begin{pmatrix} d_1(\Omega,\boldsymbol{p}) & d_2(\Omega,\boldsymbol{p}) \\ d_2(\Omega,\boldsymbol{p}) & d_3(\Omega,\boldsymbol{p}) \end{pmatrix} \\ &= \boldsymbol{N}_{21}\Omega^2 + \boldsymbol{N}_{22}(\boldsymbol{p})\Omega + \boldsymbol{N}_{23}(\boldsymbol{p}), \end{aligned} \tag{4.24}$$

where $\boldsymbol{N}_{21}$, $\boldsymbol{N}_{22}(\boldsymbol{p})$ and $\boldsymbol{N}_{23}(\boldsymbol{p})$ are defined in (1.11). The second equality in (4.24) is verified by expanding out the expressions in (4.23) and using the definition of $\boldsymbol{D}_1$, $\boldsymbol{D}_2$ and $\boldsymbol{D}_3$. This completes the verification of the linear instability criterion. The dispersion relation has the form

$$\Upsilon(\Omega,\boldsymbol{p}) = \Upsilon_2(\Omega,\boldsymbol{p}) + R_5(\Omega,\boldsymbol{p}).$$

Therefore, if there exists a root of $\Upsilon_2(\Omega,\boldsymbol{p}) = 0$ for $|\Omega|$ and $|\boldsymbol{p}|$ sufficiently small with $\boldsymbol{p} \in \boldsymbol{R}^2$ and $\mathrm{Im}(\Omega) \neq 0$, there is an unstable solution to the linear stability problem (4.5). The result is an instability theory precisely because perturbations are restricted to have small $|\boldsymbol{p}|$. That is, if $\Upsilon_2(\Omega,\boldsymbol{p}) = 0$ with $|\Omega| + |\boldsymbol{p}|$ small and $\boldsymbol{p} \in \boldsymbol{R}^2$ results in $\Omega \in \boldsymbol{R}$, the basic state is not necessarily stable as there may exist an unstable mode with a value of $\boldsymbol{p}$ outside a neighbourhood of zero.

5. Instability of small-amplitude capillary-gravity travelling waves

In this section the variational principle of § 3 and the instability criterion of § 4 will be applied to weakly nonlinear capillary-gravity waves travelling in finite depth with mean-flow effects of the form (3.1). The theory recovers all the essential features of the previous analyses. When surface tension forces are absent, the theory agrees with the instability results due to Hayes (1973) and Davey & Stewartson (1974). When surface tension is also present, the theory agrees with the results of Djordjevic & Redekopp (1977) for the instability of three-dimensional capillary-gravity travelling waves.

For small amplitude unidirectional travelling waves, we approximate a travelling wave by the following Fourier expansion:

$$\left.\begin{aligned} \eta(x,y,t) &= a_0 + a_1\cos\theta + a_2\cos 2\theta + \cdots, \\ \phi(x,y,z,t) &= b_1\frac{\cosh k(z+h)}{\cosh kh}\sin\theta + b_2\frac{\cosh 2k(z+h)}{\cosh 2kh}\cos 2\theta + \cdots, \end{aligned}\right\} \tag{5.1}$$

where $\theta = x + y - t$ and $k = \sqrt{k_1^2 + k_2^2}$. Note that ϕ is strictly periodic in (x, y, t); the mean flow effects are incorporated in expression (3.1), which generates the Lagrange multipliers arising due to the imposed constraint of mass conservation. Given expression (5.1), the other components of the vector Z, in the context of the variational principle (3.8), are

$$\left.\begin{aligned} \Phi = \phi|_{z=\eta}, \quad u = k_1\phi_x + m_1, \quad v = k_2\phi_y + m_2 \\ w_1 = k_1\eta_x/(1+k_1^2\eta_x^2+k_2^2\eta_y^2)^{1/2}, \quad w_2 = k_2\eta_y/(1+k_1^2\eta_x^2+k_2^2\eta_y^2)^{1/2}. \end{aligned}\right\} \tag{5.2}$$

The object is to determine the Fourier coefficients $a_0, a_1, \ldots, b_1, \ldots,$ using the variational principle (3.8). Although the variational principle (3.8) is global, the present analysis in terms of the Fourier approximation (5.1) is a classical weakly nonlinear analysis reinterpreted within the context of the variational principle (3.8). In particular, the direct evaluation of the elements of the sensitivity matrix $\boldsymbol{D}$ and their relation to the linear stability problem is new.

The expressions for $Z = (\Phi, \eta, w_1, w_2, \phi, u, v)$, given in (5.1), (5.2), are substituted into the seven basic functionals $\mathcal{S}$, $\overline{A}$, $\overline{\boldsymbol{B}}$, $\overline{P}$ and $\overline{\boldsymbol{Q}}$, and their explicit expressions are given in Appendix C.

The Lagrange necessary condition for a stationary point of the constrained variational principle (3.8) can now be posed in terms of the Fourier coefficients of ϕ and η. Let

$$\mathcal{F}(a_0, a_1, \ldots, b_1, \ldots; \omega, \boldsymbol{k}, \gamma, \boldsymbol{m}) = \mathcal{S} - \omega\overline{A} - \boldsymbol{k}\cdot\overline{\boldsymbol{B}} - \gamma\overline{P} - \boldsymbol{m}\cdot\overline{\boldsymbol{Q}}, \tag{5.3}$$

with $\mathcal{S}$, $\overline{A}$, $\overline{\boldsymbol{B}}$, $\overline{P}$ and $\overline{\boldsymbol{Q}}$ given in Appendix C. The Fourier coefficients $a_0, a_1, \ldots, b_1, b_2, \ldots$ are obtained as functions of $\omega, \boldsymbol{k}, \gamma, \boldsymbol{m}$ by solving the system of algebraic equations

$$\frac{\partial \mathcal{F}}{\partial a_0} = \frac{\partial \mathcal{F}}{\partial a_j} = \frac{\partial \mathcal{F}}{\partial b_j} = 0, \quad \text{for } j = 1, 2, \ldots.$$

We find

$$\begin{aligned} b_1 &= \frac{(\omega - \boldsymbol{k}\cdot\boldsymbol{m})}{tk}\left\{1 - \frac{k}{t}a_0 - k\left(\frac{t}{2}+\frac{1}{t}\right)a_2 + \frac{k^2}{8t^2}(2-t^2)a_1^2\right\}a_1 + \cdots, \\ b_2 &= \frac{1}{kt_2}(\omega - \boldsymbol{k}\cdot\boldsymbol{m})\left(a_2 - \frac{k}{2t}a_1^2\right) + \cdots, \\ a_2 &= \frac{(\omega - \boldsymbol{k}\cdot\boldsymbol{m})^2}{4gt^2}\left(\frac{3-t^2}{t^2 - \widetilde{T}(3-t^2)}\right)a_1^2 + \cdots, \\ a_0 &= -\frac{1}{g}(\gamma + \tfrac{1}{2}\boldsymbol{m}\cdot\boldsymbol{m}) - \frac{(\omega - \boldsymbol{k}\cdot\boldsymbol{m})^2}{4gt^2}(1-t^2)\,a_1^2 + \cdots, \end{aligned}$$

where

$$t = \tanh kh, \quad t_2 = \tanh 2kh \quad \text{and} \quad \widetilde{T} = \sigma k^2/g.$$

Finally, solving $\partial\mathcal{F}/\partial a_1 = 0$ results in

$$\xi_2 k^2 a_1^2 = \frac{(\omega - \boldsymbol{m}\cdot\boldsymbol{k})^2}{gkt} - (1+\widetilde{T}) + (1+\widetilde{T})\frac{k}{gt}(1-t^2)(\gamma + \tfrac{1}{2}\boldsymbol{m}\cdot\boldsymbol{m}) + \cdots, \tag{5.4}$$

where

$$\xi_2 = \xi_3 - \frac{(1+\widetilde{T})^2}{4t^2}(1-t^2)^2 \tag{5.5}$$

and

$$\xi_3 = -\frac{3}{8}\widetilde{T} + \frac{(1+\widetilde{T})}{2t^2}(2t^2-1) + \frac{(1+\widetilde{T})^2(3-t^2)^2}{8t^2(t^2-\widetilde{T}(3-t^2))}. \tag{5.6}$$

The expression for ξ_3 recovers the first part of the ν-coefficient in the modulation equation of Djordjevic & Redekopp (1977, equation (2.17)). The precise relation between ξ_3 and $\widetilde{\nu}$ of Djordjevic & Redekopp is

$$\xi_3 = (1+\widetilde{T})\frac{\omega}{2t^2k^2}\widetilde{\nu}.$$

The coefficient $\widetilde{\nu}$ is the part of ν in equation (2.17) of Djordjevic & Redekopp without the $(gh-c_g^2)^{-1}$ term. In the present formulation, the $(gh-c_g^2)^{-1}$ term will show up after the derivatives in the sensitivity matrix are constructed. This is an indication of how the present formulation acquires the information necessary for the instability theory, in comparison to the way modulation equations acquire the same information.

However, although expression (5.4) is reminiscent of the nonlinear dispersion relation for weakly nonlinear three-dimensional capillary-gravity travelling waves with mean flow effects, it is, within the context of the variational principle (3.8), an intermediate step. In the variational principle (3.8), the above expressions for the coefficients $a_0, a_1, \ldots, b_1, \ldots$ are functions of the Lagrange multipliers $(\omega, \boldsymbol{k}, \gamma, \boldsymbol{m})$ (and parameters) and values for the Lagrange multipliers are determined by substituting the coefficients $a_0, a_1, \ldots, b_1, \ldots$ into the constraints. Also important for the present theory is that the Lagrange functional $\mathcal{F}$ is decomposed in such a way that stability information can be extracted.

Substitution of the above approximate expression for the family of travelling waves into the six constraint sets results in

$$\overline{A} = \frac{1}{2}\left(\frac{\omega - \boldsymbol{m}\cdot\boldsymbol{k}}{tk}\right)\left[-1 - \frac{k}{gt}(1-t^2)(\gamma + \tfrac{1}{2}\boldsymbol{m}\cdot\boldsymbol{m})\right]a_1^2 + \cdots, \tag{5.7}$$

$$\overline{\boldsymbol{B}} = \frac{1}{2}\left[\frac{\boldsymbol{m}}{kt}(\omega - \boldsymbol{m}\cdot\boldsymbol{k}) + \frac{1}{2}\frac{\boldsymbol{k}}{k}(t + kh(1-t^2))\frac{(\omega-\boldsymbol{m}\cdot\boldsymbol{k})^2}{t^2k^2} + \sigma\boldsymbol{k}\right]a_1^2 + \cdots, \tag{5.8}$$

$$\overline{P} = -\frac{1}{g}(\gamma + \tfrac{1}{2}\boldsymbol{m}\cdot\boldsymbol{m}) - \frac{1}{4g}\left(\frac{1-t^2}{t^2}\right)(\omega - \boldsymbol{m}\cdot\boldsymbol{k})^2 a_1^2 + \cdots, \tag{5.9}$$

$$\overline{\boldsymbol{Q}} = \boldsymbol{m}h + \frac{(\omega - \boldsymbol{m}\cdot\boldsymbol{k})}{2t}\frac{\boldsymbol{k}}{k}a_1^2 + \cdots, \tag{5.10}$$

where a_1^2 here is considered as a function of the parameters $\omega, \boldsymbol{k}, \gamma, \boldsymbol{m}$ through (5.4). The constraint sets $\overline{A}, \overline{\boldsymbol{B}}, \overline{P}, \overline{\boldsymbol{Q}}$ have been reduced to functions of the six Lagrange multipliers $\omega, \boldsymbol{k}, \gamma, \boldsymbol{m}$ (and parameters).

In the context of the variational principle (3.8), the values of $(\overline{A}, \overline{\boldsymbol{B}}, \overline{P}, \overline{\boldsymbol{Q}})$ are fixed at level sets $(I_1, \ldots, I_6)$ and this determines the values of the elements $(\omega, \boldsymbol{k}, \gamma, \boldsymbol{m})$. We are interested here in applying the linear instability criterion of § 4, and therefore the explicit inversion of (5.10), to obtain the functional dependence of $(\omega, \boldsymbol{k}, \gamma, \boldsymbol{m})$ on $(I_1, \ldots, I_6)$ is not needed. The linear instability criterion requires the elements of the Jacobian matrix $\boldsymbol{D}$. The calculation of the elements in $\boldsymbol{D}$ is straightforward using (5.10) and (5.4) and we record the result here. For simplicity, we will suppose that $(I_1, \ldots, I_6)$ are adjusted so that $\gamma = \boldsymbol{m} = 0$ when $I = 0$. This choice assumes that the only mean flow is that generated by the waves. Note, however, that it is important to take the derivatives with respect to γ and $\boldsymbol{m}$ and form the entries of $\boldsymbol{D}$ before specifying the basic state and setting $\gamma = \boldsymbol{m} = 0$.

Partitioning the matrix $\boldsymbol{D}$ as in (4.22), the 3×3 submatrices have the following expansions:

$$\boldsymbol{D}_1 = \begin{pmatrix} \overline{A}_\omega & \overline{A}_{\boldsymbol{k}} \\ \overline{\boldsymbol{B}}_\omega & \overline{\boldsymbol{B}}_{\boldsymbol{k}} \end{pmatrix} = \frac{1+\widetilde{T}}{\xi_2 k^2}\frac{1}{tk}\begin{pmatrix} -1 & |\boldsymbol{c_g}|\boldsymbol{k}/k \\ |\boldsymbol{c_g}|\boldsymbol{k}^{\mathrm{T}}/k & -|\boldsymbol{c_g}|^2\boldsymbol{k}\boldsymbol{k}^{\mathrm{T}}/k^2 \end{pmatrix}$$
$$+\frac{a_1^2}{2tk}\begin{pmatrix} -1 & \dfrac{\omega}{tk^2}\boldsymbol{k}(t+k\dot{t}) \\ \dfrac{\omega}{tk^2}\boldsymbol{k}^{\mathrm{T}}(t+k\dot{t}) & \omega\,\mathrm{Hess}_{\boldsymbol{k}}(\omega) - \dfrac{3}{k^2}|\boldsymbol{c_g}|^2\boldsymbol{k}\boldsymbol{k}^{\mathrm{T}} + 4\dfrac{\sigma t}{\omega}|\boldsymbol{c_g}|\boldsymbol{k}\boldsymbol{k}^{\mathrm{T}} \end{pmatrix} + \cdots, \tag{5.11}$$

$$\boldsymbol{D}_2 = \begin{pmatrix} \overline{A}_\gamma & \overline{A}_{\boldsymbol{m}} \\ \overline{\boldsymbol{B}}_\gamma & \overline{\boldsymbol{B}}_{\boldsymbol{m}} \end{pmatrix}$$
$$= \frac{1+\widetilde{T}}{\xi_2 k^2}\frac{1}{t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -|\boldsymbol{c_g}|k_1 & 0 \\ 0 & 0 & -|\boldsymbol{c_g}|k_2 \end{bmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{bmatrix} -\omega\dot{t}/(2ght) & 0 & 0 \\ 0 & k_1/k & 0 \\ 0 & 0 & k_2/k \end{bmatrix}$$
$$+\cdots, \tag{5.12}$$

$$\boldsymbol{D}_3 = \begin{pmatrix} \overline{P}_\gamma & \overline{P}_{\boldsymbol{m}} \\ \overline{\boldsymbol{Q}}_\gamma & \overline{\boldsymbol{Q}}_{\boldsymbol{m}} \end{pmatrix} = -\frac{1}{g}\begin{pmatrix} 1 & m_1 & m_2 \\ m_1 & -gh & 0 \\ m_2 & 0 & -gh \end{pmatrix}$$
$$-\frac{(1+\widetilde{T})^2}{\xi_2 k^2}\begin{pmatrix} -\dfrac{k^2\dot{t}^2}{4gh^2t^2} & \dfrac{k\dot{t}}{2\omega th}\boldsymbol{k} \\ \dfrac{k\dot{t}}{2\omega th}\boldsymbol{k}^{\mathrm{T}} & -\dfrac{1}{kt(1+\widetilde{T})}\boldsymbol{k}\boldsymbol{k}^{\mathrm{T}} \end{pmatrix} + \cdots. \tag{5.13}$$

The dot over t indicates a derivative with respect to k: $\dot{t} = h(1-t^2)$ and $\boldsymbol{c_g}$ is the vector-valued group velocity of the linear wave (cf. Appendix D). Some of the results used in the construction of $\boldsymbol{D}$ are as follows. The derivatives of a_1^2 are needed and are obtained by differentiating (5.4), resulting in

$$\frac{\partial}{\partial\omega}a_1^2 = \frac{1+\widetilde{T}}{\xi_2 k^2}\left(\frac{2}{\omega}\right) + \cdots, \quad \frac{\partial}{\partial\boldsymbol{k}}a_1^2 = \frac{1+\widetilde{T}}{\xi_2 k^2}\left(-\frac{2}{\omega}\boldsymbol{c_g}\right) + \cdots,$$
$$\frac{\partial}{\partial\gamma}a_1^2 = \frac{1+\widetilde{T}}{\xi_2 k^2}\left(\frac{k}{gt}(1-t^2)\right) + \cdots, \quad \frac{\partial}{\partial\boldsymbol{m}}a_1^2 = \frac{1+\widetilde{T}}{\xi_2 k^2}\left(-\frac{2\boldsymbol{k}}{\omega}\right) + \cdots.$$

The construction of $\boldsymbol{D}_1$ requires derivatives of the vector-valued group velocity $\boldsymbol{c_g}$ and the calculation is given in Appendix D (the definition of $\mathrm{Hess}_{\boldsymbol{k}}(\omega)$ is in equation (D 6)). The construction of $\overline{\boldsymbol{B}}_{\boldsymbol{k}}$ is the lengthiest calculation and it is simplified as follows. Using (5.10), $\overline{\boldsymbol{B}}$ can be written in the form

$$\overline{\boldsymbol{B}} = \left(\frac{\boldsymbol{m}}{2tk}(\omega - \boldsymbol{m}\cdot\boldsymbol{k}) + \boldsymbol{f}\right)a_1^2 + \cdots,$$

where $\boldsymbol{f}$ is defined in Appendix D (equation (D 8)). Therefore, the Jacobian $\overline{\boldsymbol{B}}_{\boldsymbol{k}}$ is

$$\overline{\boldsymbol{B}}_{\boldsymbol{k}} \stackrel{\text{def}}{=} \mathrm{Jac}_{\boldsymbol{k}}(\overline{\boldsymbol{B}}) = \mathrm{Jac}_{\boldsymbol{k}}(\boldsymbol{f})a_1^2 - \frac{2}{\omega}\frac{1+\widetilde{T}}{\xi_2 k^2}|\boldsymbol{c_g}|\boldsymbol{f}\cdot\boldsymbol{k}^{\mathrm{T}} + \cdots.$$

Then, using the expression for $\mathrm{Jac}_{\boldsymbol{k}}(\boldsymbol{f})$ given in (D 9) results in the appropriate term in $\boldsymbol{D}_1$.

Note that γ and $\boldsymbol{m}$ have been set to zero after the derivatives have been formed; except in the first term in $\boldsymbol{D}_3$. When $\boldsymbol{m} \neq 0$ there is a basic wave-independent mean flow generated by a suitable choice of the mass flux. Here we will take the wave-independent mean flow to be zero, but first a point will be made about perturbations of the mean-flow. The matrix $\boldsymbol{D}_2$ gives the interaction terms between the waves and the mean flow, whereas $\boldsymbol{D}_3$ contains the pure mean flow terms. The expression

$$d_3(\Omega,\boldsymbol{p}) = \left\langle \begin{pmatrix} -\Omega \\ \boldsymbol{p} \end{pmatrix}, \boldsymbol{D}_3 \begin{pmatrix} -\Omega \\ \boldsymbol{p} \end{pmatrix} \right\rangle_{\boldsymbol{R}^3} \tag{5.14}$$

then gives the eigenvalues $\{\Omega\}$ for perturbations of the uniform flow when the perturbations are of the form $\exp[\mathrm{i}(p_1 x + p_2 y - \Omega t)]$. The $\boldsymbol{R}^3$ subscript on the above inner product indicates that no complex conjugation is involved in the product.

Writing out (5.14) using (5.13) results in

$$d_3(\Omega,\boldsymbol{p}) = -(1/g)(\Omega^2 - 2\Omega\boldsymbol{p}\cdot\boldsymbol{m} - gh\boldsymbol{p}\cdot\boldsymbol{p}) + \cdots,$$

which when $|\boldsymbol{m}| \ll 1$ results in the modes

$$\Omega = \boldsymbol{p}\cdot\boldsymbol{m} \pm |\boldsymbol{p}|\sqrt{gh} + \cdots \tag{5.15}$$

where the higher-order terms include terms of order $|\boldsymbol{m}|^2$ and the neglected terms in $\boldsymbol{D}_3$ due to wave-induced mean flow. To leading order, expression (5.15) is the perturbation of the uniform flow that one would obtain from the shallow water equations (cf. Appendix B, equation (B 6)). However, hereafter we suppose that the wave-independent mean velocity vanishes and set $\boldsymbol{m} = 0$ in the first term in $\boldsymbol{D}_3$.

Define $d_j(\Omega,\boldsymbol{p})$ for $j = 1,2,3$ as in equation (4.23). Then by the instability criterion of § 4, a wave is unstable if, for fixed $\boldsymbol{D}$ there exists a solution of

$$\Upsilon_2(\Omega,\boldsymbol{p}) = d_1(\Omega,\boldsymbol{p})d_3(\Omega,\boldsymbol{p}) - d_2(\Omega,\boldsymbol{p})^2 = 0,$$

with $\boldsymbol{p} \in \boldsymbol{R}^2$ and sufficiently small and $\mathrm{Im}(\Omega) \neq 0$. To find such a solution we first note that when $a_1 = 0$ the functions d_1, d_2 and d_3 reduce to

$$d_1 = -\frac{1+\widetilde{T}}{\xi_2 k^2}\frac{1}{tk}(\Omega - \boldsymbol{c_g}\cdot\boldsymbol{p})^2,$$

$$d_2 = \frac{1+\widetilde{T}}{\xi_2 k^2}\frac{1}{t}(\Omega - \boldsymbol{c_g}\cdot\boldsymbol{p})\left(\frac{\omega\dot{t}}{2ght}\Omega + \frac{\boldsymbol{k}\cdot\boldsymbol{p}}{k}\right),$$

$$d_3 = -\frac{1}{g}(\Omega^2 - gh\boldsymbol{p}\cdot\boldsymbol{p}) - \frac{1+\widetilde{T}}{\xi_2 k^2}\frac{k}{t}\left(\frac{\omega\dot{t}}{2ght}\Omega + \frac{\boldsymbol{k}\cdot\boldsymbol{p}}{k}\right)^2.$$

Therefore, when $a_1 = 0$, $\Upsilon_2(\Omega,\boldsymbol{p})$ reduces to

$$\begin{aligned}\Upsilon_2(\Omega,\boldsymbol{p}) &= d_1(\Omega,\boldsymbol{p})d_3(\Omega,\boldsymbol{p}) - d_2(\Omega,\boldsymbol{p})^2 \\ &= \frac{1+\widetilde{T}}{\xi_2 k^2}\frac{1}{gtk}(\Omega - \boldsymbol{c_g}\cdot\boldsymbol{p})^2(\Omega^2 - gh\boldsymbol{p}\cdot\boldsymbol{p}),\end{aligned}$$

with solutions

$$\Omega = \left\{ |\boldsymbol{c_g}| \frac{\boldsymbol{k} \cdot \boldsymbol{p}}{k}, \pm |\boldsymbol{p}| \sqrt{gh} \right\}. \tag{5.16}$$

The eigenvalues $\pm|\boldsymbol{p}|\sqrt{gh}$ correspond to perturbations of the uniform flow (cf. equation (5.15) and Appendix B). The eigenvalue $\boldsymbol{c_g} \cdot \boldsymbol{p}/k$ has multiplicity two as long as

$$gh - \mu^2 |\boldsymbol{c_g}|^2 \neq 0, \tag{5.17}$$

where

$$\mu = \frac{\boldsymbol{k} \cdot \boldsymbol{n}}{k} \quad \text{and} \quad \boldsymbol{n} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|}. \tag{5.18}$$

We will suppose that (5.17) is satisfied, although it should be noted that there exist critical values of the parameters at which (5.17) is violated. The singularity $gh = \mu^2 |\boldsymbol{c_g}|^2$ is called the long-wave short-wave resonance (cf. Djordjevic & Redekopp 1977, § 4).

For $|a_1|$ small, we will consider the instability associated with the splitting of the double eigenvalue $\boldsymbol{c_g} \cdot \boldsymbol{p}/k$. Let

$$\Omega = |\boldsymbol{c_g}| \frac{\boldsymbol{k} \cdot \boldsymbol{p}}{k} + a_1 \Omega_1 \frac{\boldsymbol{k} \cdot \boldsymbol{p}}{k} + \cdots. \tag{5.19}$$

Substitution of the above expression for Ω into d_1, d_2 and d_3 results in

$$d_1 = \frac{1}{tk} \left(\frac{1}{2} \frac{\omega}{\mu^2} \langle \boldsymbol{n}, \mathrm{Hess}_{\boldsymbol{k}}(\omega) \boldsymbol{n} \rangle - \frac{1 + \widetilde{T}}{\xi_2 k^2} \Omega_1^2 \right) \left(\frac{\boldsymbol{k} \cdot \boldsymbol{p}}{k} \right)^2 a_1^2 + \cdots,$$

$$d_2 = \frac{1 + \widetilde{T}}{\xi_2 k^2} \frac{1}{t} a_1 \Omega_1 \left(1 + \frac{\omega \dot{t}}{2ght} |\boldsymbol{c_g}| \right) \left(\frac{\boldsymbol{k} \cdot \boldsymbol{p}}{k} \right)^2 + \cdots,$$

$$d_3 = \left(\frac{\boldsymbol{k} \cdot \boldsymbol{p}}{k} \right)^2 \left[\frac{1}{g} \left(\frac{gh}{\mu^2} - |\boldsymbol{c_g}|^2 \right) - \frac{1 + \widetilde{T}}{\xi_2 k^2} \frac{k}{t} \left(1 + \frac{\omega \dot{t}}{2ght} |\boldsymbol{c_g}| \right)^2 \right] + \cdots,$$

where $\mathrm{Hess}_{\boldsymbol{k}}(\omega)$ is defined in Appendix D.

Therefore, the stability function $\Upsilon_2(\Omega, \boldsymbol{p})$ becomes

$$\Upsilon_2(\Omega, \boldsymbol{p}) = -a_1^2 \frac{1 + \widetilde{T}}{\xi_2 k^2} \left(\frac{\boldsymbol{k} \cdot \boldsymbol{p}}{k} \right)^4 \left\{ \left(\frac{gh}{\mu^2} - |\boldsymbol{c_g}|^2 \right) \frac{\Omega_1^2}{gtk} - \tfrac{1}{2} d_1^{(2)} \frac{\xi_2 k}{gt(1 + \widetilde{T})} \left(\frac{gh}{\mu^2} - |\boldsymbol{c_g}|^2 \right) + \frac{1}{2} \frac{d_1^{(2)}}{t^2} \left(1 + \frac{\omega \dot{t}}{2ght} |\boldsymbol{c_g}| \right)^2 \right\} + \cdots,$$

where

$$d_1^{(2)} = \frac{\omega}{\mu^2} \langle \boldsymbol{n}, \mathrm{Hess}_{\boldsymbol{k}}(\omega) \boldsymbol{n} \rangle.$$

Setting $\Upsilon_2(\Omega, \boldsymbol{p})$ to zero, under the following non-degeneracy conditions:

$$\begin{aligned} a_1^2 &\neq 0 && \text{(non-zero but small amplitude)},\\ \xi_2 &\neq 0 && \text{(critical coefficient in (5.4) non-zero)},\\ \boldsymbol{p} &\neq 0 && \text{(non-zero but small wavenumber sideband)},\\ gh - \mu^2|\boldsymbol{c_g}|^2 &\neq 0 && \text{(long-wave short-wave resonance excluded)},\\ \langle \boldsymbol{n}, \mathrm{Hess}_{\boldsymbol{k}}(\omega)\boldsymbol{n}\rangle &\neq 0 && \text{(non-degenerate group-velocity Jacobian)},\\ \nu(kh,\mu) &\neq 0 && \text{(critical coefficient introduced below)}, \end{aligned}$$

results in the following expression for the term Ω_1:

$$\Omega_1^2 = \tfrac{1}{2} d_1^{(2)} \frac{\xi_2 k^2}{(1+\widetilde{T})} - \tfrac{1}{2}\frac{d_1^{(2)}}{t^2}\frac{gtk}{((gh/\mu^2) - |\boldsymbol{c_g}|^2)}\left(1 + \frac{\omega\dot{t}}{2ght}|\boldsymbol{c_g}|\right)^2 \tag{5.20}$$

$$= \frac{\omega k}{2ht\mu^2}\langle \boldsymbol{n}, \mathrm{Hess}_{\boldsymbol{k}}(\omega)\boldsymbol{n}\rangle \left[\frac{kht\xi_2}{(1+\widetilde{T})} - \frac{gh}{gh-\mu^2|\boldsymbol{c_g}|^2}\left(1+\frac{\omega\dot{t}}{2ght}|\boldsymbol{c_g}|\right)^2\mu^2\right] \tag{5.21}$$

$$= \frac{\omega k}{2ht\mu^2}\langle \boldsymbol{n}, \mathrm{Hess}_{\boldsymbol{k}}(\omega)\boldsymbol{n}\rangle \nu(kh,\mu), \tag{5.22}$$

where

$$\nu(kh,\mu) = \frac{kht}{(1+\widetilde{T})}\xi_3 - \frac{h}{t}\left[\frac{gt\mu^2 + \omega\mu^2(1-t^2)|\boldsymbol{c_g}| + \frac{1}{4}rgkh(1+\widetilde{T})(1-t^2)^2}{gh-\mu^2|\boldsymbol{c_g}|^2}\right] \tag{5.23}$$

with ξ_3 given in equation (5.6). When $\Omega_1^2 < 0$, or

$$\langle \boldsymbol{n}, \mathrm{Hess}_{\boldsymbol{k}}(\omega)\boldsymbol{n}\rangle \nu(kh,\mu) < 0, \tag{5.24}$$

the basic state is unstable, for a_1^2 sufficiently small, as then there exists a solution of $\Upsilon_2(\Omega, \boldsymbol{p}) = 0$ with $\mathrm{Im}(\Omega) \neq 0$.

We now compare the result (5.24) with existing results in the literature. First, setting $\widetilde{T} = 0$, we find

$$\frac{k^3\nu}{kh} = k^3\left(\frac{9-10t^2+9t^4}{8t^3}\right) - \frac{g}{gh-\mu^2|\boldsymbol{c_g}|^2}\left[\mu^2k^2 + \frac{k^3\mu^2}{\omega}|\boldsymbol{c_g}|(1-t^2) + \frac{hk^3}{4t}(1-t^2)^2\right]. \tag{5.25}$$

The right-hand side of (5.25) is precisely $\Omega_A{}'$ of Hayes (1973, equation (6.10)) and his instability criterion is

$$\Omega_A{}' H_{kk} : \boldsymbol{nn} < 0, \tag{5.26}$$

which is seen to agree with that in (5.24) upon noting that $H_{kk} : \boldsymbol{nn}$ in Hayes notation has the same sign as $\langle \boldsymbol{n}, \mathrm{Hess}_{\boldsymbol{k}}(\omega)\boldsymbol{n}\rangle$ in the present notation. Hayes (1973, figure 1) plots zeros of the two factors in (5.26) for pure gravity waves indicating regions of instability. The above result also agrees with the instability results of Davey & Stewartson (1974).

When surface tension is present the above result agrees with the instability results obtained using a modulation equation by Djordjevic & Redekopp (1977). To correspond with the notation of Djordjevic & Redekopp, let $c_p = \frac{\omega}{k}$, set $\mu = 1$ and consider the group velocity to be oriented in the k_1 direction. Then ν in (5.23) can be written in the form

$$\nu = \frac{kht}{(1+\widetilde{T})}\xi_3 - \frac{k^3h}{t(1+\widetilde{T})}\left[\frac{c_p^2 + c_p|\boldsymbol{c_g}|(1+\widetilde{T})(1-t^2) + \frac{1}{4}rgh(1-t^2)^2(1+\widetilde{T})^2}{gh - |\boldsymbol{c_g}|^2}\right]$$
$$= \frac{\omega h}{2tk^3}\nu_{\mathrm{DR}},$$

after substituting in the expression for ξ_3. The variable ν_{DR} is ν of Djordjevic & Redekopp (1977, equation (2.17)) (after a typographical error in Djordjevic & Redekopp is corrected; the required $\frac{1}{4}$ in the numerator is missing). In Djordjevic & Redekopp, $\mu = 1$ since the coefficients in their modulation equation are evaluated in the two-dimensional state. Djordjevic & Redekopp plot zeros of ν_{DR} (as well as other critical coefficients) in the kh versus $\widetilde{T}$ plane.

This completes the comparison of the result obtained using the variational principle of § 3 and the instability theory of § 4 with the classical results. The advantage of the present theory is that the analysis of the instability problem is done directly, relating the sign of the exact linear stability exponent to properties given by the exact variational principle of § 3.

Finally, we note that the form of the eigenfunction of the unstable solution is given and can be expressed as a sum of the derivatives of the basic state with respect to $I_1, \ldots, I_6$ (cf. § 4, equations (4.8) and (4.14)).

6. Concluding remarks

A framework for the analysis and linear instability of periodic patterns on the ocean surface including mean-flow interactions has been presented. The framework was based on a generalized Hamiltonian formulation which assigns a distinct symplectic operator for each unbounded space direction as well as time. The theory was used to: (a) give a new characterization of mean-flow dynamics; (b) derive a variational principle for space- and time-periodic states with mean-flow interactions; and (c) to derive an instability criterion for periodic three-dimensional travelling waves with a wave-generated meanflow. As an example, the instability theory was applied to weakly nonlinear Stokes waves in finite depth recovering all the previous results on instability by Benney-Roskes, Hayes, Davey-Stewartson and Djordjevic & Redekopp. The theory is generalizable to other water-wave problems (for example, interfacial waves and oceanographic rotating flows) as well as wave problems in other physical applications.

The variational principle of § 3 is applicable to all space-time periodic patterns and therefore should be important for classifying periodic waves on the ocean surface. The classification of periodic patterns on the ocean surface begins with the group-theoretic question: what are all possible periodic tilings of a two-dimensional plane? The classification of such 'wallpaper patterns' is a classic problem in group theory (cf. Armstrong 1988, ch. 25, 26). It is known that all possible periodic tilings are among the following five lattice types: rectangular, rhombic, centred rectangular, square and hexagonal, and when subclassified by point groups there are exactly 17 distinct symmetry groups. In all cases the patterns are doubly periodic and therefore will fit into the framework and variational principle of § 3. A complete classification would require studying the existence and linear stability of each of the possible periodic patterns. Since the number of distinct lattice types is finite, such a classification is, in principle, tractable. For example, once the lattice type is fixed, it is straightforward to use the variational principle of § 3, either for a weakly nonlinear analysis or numerical calculations, to construct branches of such waves. It appears that the only families of periodic patterns for the ocean surface that have heretofore been calculated are the square and rectangular patterns. Whether rhombic or hexagonal patterns are important for ocean wave dynamics is not known. Naturally, the importance of such patterns will depend on their linear stability properties.

To study the instability of arbitrary periodic patterns on the ocean surface, the analysis of § 4 would require some modification. For an arbitrary periodic pattern, given by the variational principle of § 3, the basic state (4.1) would be replaced by

$$\{\widehat{Z}(x,y,t;I),\omega(I),\boldsymbol{k}(I),\gamma(I),\boldsymbol{m}(I):I=(I_1,\ldots,I_6)\}.$$

For a travelling wave there exists a frame of reference so that $\widehat{Z}$ depends only on the combination $\theta = x + y - t$, whereas in the general case the dependence on x, y and t is independent. Formally, the linear stability problem is again given by (4.5). However, the kernel of $\mathcal{L}_0$ will be larger:

$$\mathrm{Ker}(\mathcal{L}_0)=\{\widehat{Z}_x,\widehat{Z}_y,\widehat{Z}_t,V\},$$

where $\widehat{Z}_x$, $\widehat{Z}_y$ and $\widehat{Z}_t$ are independent. Therefore there will be four solvability conditions:

$$\left.\begin{aligned}
[\widehat{Z}_x,-\mathrm{i}\Omega\boldsymbol{M}U+\mathrm{i}p_1\boldsymbol{K}(\widehat{\boldsymbol{u}})U+\mathrm{i}p_2\boldsymbol{L}(\widehat{\boldsymbol{v}})U]=0,\\
[\widehat{Z}_y,-\mathrm{i}\Omega\boldsymbol{M}U+\mathrm{i}p_1\boldsymbol{K}(\widehat{\boldsymbol{u}})U+\mathrm{i}p_2\boldsymbol{L}(\widehat{\boldsymbol{v}})U]=0,\\
[\widehat{Z}_t,-\mathrm{i}\Omega\boldsymbol{M}U+\mathrm{i}p_1\boldsymbol{K}(\widehat{\boldsymbol{u}})U+\mathrm{i}p_2\boldsymbol{L}(\widehat{\boldsymbol{v}})U]=0,\\
[\widehat{V},-\mathrm{i}\Omega\boldsymbol{M}U+\mathrm{i}p_1\boldsymbol{K}(\widehat{\boldsymbol{u}})U+\mathrm{i}p_2\boldsymbol{L}(\widehat{\boldsymbol{v}})U]=0,
\end{aligned}\right\}\tag{6.1}$$

with U taking the general form

$$U=\alpha_1\widehat{Z}_x+\alpha_2\widehat{Z}_y+\alpha_3\widehat{Z}_t+\alpha_4V+W\qquad W|_{\Omega=\boldsymbol{p}=0}=0.$$

Since W will depend linearly on $\alpha\in\boldsymbol{C}^4$, the set (6.1) has the general form

$$[\boldsymbol{N}(\Omega,\boldsymbol{p})]\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_4\end{pmatrix}=0,$$

where $\boldsymbol{N}(\Omega,\boldsymbol{p})$ is now a 4×4 matrix with, in general, complex-valued entries. We can conjecture that, as in § 4, the terms of $\boldsymbol{N}(\Omega,\boldsymbol{p})$ linear in Ω and $\boldsymbol{p}$ will vanish, resulting in a dispersion relation of the form

$$\Upsilon(\Omega,\boldsymbol{p})=\det[\boldsymbol{N}_2(\Omega,\boldsymbol{p})]+R_9(\Omega,\boldsymbol{p}).\tag{6.2}$$

A basic state is linearly unstable if there exists a root of $\Upsilon_2(\Omega,\boldsymbol{p})=0$ with $|\Omega|+|\boldsymbol{p}|$ sufficiently small, $\boldsymbol{p}\in\boldsymbol{R}^2$ and $\mathrm{Im}(\Omega)\neq0$. The leading part of the dispersion relation, $\Upsilon_2(\Omega,\boldsymbol{p})$, is significantly more complicated than that for the travelling wave given in § 4. Setting $\Upsilon_2(\Omega,\boldsymbol{p})=0$ results in a homogeneous polynomial of degree eight in Ω and $\boldsymbol{p}$ and will therefore contain additional types of instabilities.

The linear stability problem for arbitrary periodic patterns on the ocean surface has not been previously considered nor formulated. Therefore, the programme sketched above, using the variational principle of § 3 to construct families of periodic patterns with rhombic, hexagonal, centred rectangular, square and rectangular lattices and development of a global instability criterion following the framework of § 4 and the discussion above, would provide significant new information about the general pattern formation question for periodic waves on the ocean surface in infinite as well as finite-depth fluids.

Appendix A. Uniform flows—a variational principle

Uniform flows are states that are independent of x, y and t. In terms of the formulation (3.5), it follows that uniform flows satisfy

$$\nabla S(Z) - \gamma \nabla P(Z) - m_1 \nabla Q_1(Z) - m_2 \nabla Q_2(Z) = 0. \tag{A 1}$$

Define

$$\widehat{S} = S - \gamma P,$$

and suppose γ is fixed. Then uniform flows have the following variational characterization: *uniform flows correspond, for given γ, to critical points of $\widehat{S}$ on level sets of Q_1 and Q_2.*

The Lagrange necessary condition for the above variational principle is then (A 1) with γ fixed. Using the definitions for the functionals S, P, Q_1 and Q_2 given in § 2, the solution of (A 1) is

$$u = m_1, \quad v = m_2 \quad \text{and} \quad \eta \stackrel{\text{def}}{=} h = -(1/g)(\gamma + \tfrac{1}{2}(m_1^2 + m_2^2)). \tag{A 2}$$

Several interesting facts follow from this formulation. First note that

$$\mathrm{Jac}_{\boldsymbol{m}}(\boldsymbol{Q}) \stackrel{\text{def}}{=} \begin{pmatrix} \dfrac{\partial Q_1}{\partial m_1} & \dfrac{\partial Q_1}{\partial m_2} \\ \dfrac{\partial Q_2}{\partial m_1} & \dfrac{\partial Q_2}{\partial m_2} \end{pmatrix} = \begin{pmatrix} h - \dfrac{m_1^2}{g} & -\dfrac{m_1 m_2}{g} \\ -\dfrac{m_1 m_2}{g} & h - \dfrac{m_2^2}{g} \end{pmatrix}$$

and

$$\det \mathrm{Jac}_{\boldsymbol{m}}(\boldsymbol{Q}) = -h^2 \left(\frac{m_1^2 + m_2^2}{gh} - 1 \right) = -h^2(F^2 - 1), \tag{A 3}$$

where F is, by definition, a Froude number. In other words, critical flow occurs when $\det(Jac_{\boldsymbol{m}}(\boldsymbol{Q})) = 0$. From the Lagrange multiplier theory, we arrive at related identities. It is elementary to show that

$$m_j = \frac{\partial \widehat{S}}{\partial Q_j}, \quad j = 1, 2,$$

and so

$$\mathrm{Hess}_Q(\widehat{S}) = \begin{pmatrix} \dfrac{\partial^2 \widehat{S}}{\partial Q_1^2} & \dfrac{\partial^2 \widehat{S}}{\partial Q_1 \partial Q_2} \\ \dfrac{\partial^2 \widehat{S}}{\partial Q_2 \partial Q_1} & \dfrac{\partial^2 \widehat{S}}{\partial Q_2^2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial m_1}{\partial Q_1} & \dfrac{\partial m_1}{\partial Q_2} \\ \dfrac{\partial m_2}{\partial Q_1} & \dfrac{\partial m_2}{\partial Q_2} \end{pmatrix}$$

and since $\mathrm{Jac}_Q(\boldsymbol{m}) = [\mathrm{Jac}_{\boldsymbol{m}}(\boldsymbol{Q})]^{-1}$, it follows that critical flows occur when $\det(\mathrm{Hess}_Q(\widehat{S}))$ is singular. The functionals $\widehat{S}$ and $\boldsymbol{Q}$, when evaluated on the state

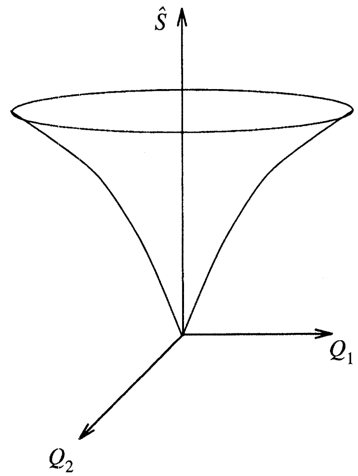

Figure 1. Parametrization of the surface (A 4) in the $(\widehat{S}, \boldsymbol{Q})$ space.

(A 2), become

$$\widehat{S} = \tfrac{1}{2}gh^2 + h(m_1^2 + m_2^2), \quad Q_1 = m_1 h, \quad Q_2 = m_2 h, \tag{A 4}$$

where

$$h = -(1/g)(\gamma + \tfrac{1}{2}(m_1^2 + m_2^2)).$$

For any fixed $\gamma < 0$, the map $(h, m_1, m_2) \mapsto (\widehat{S}, \boldsymbol{Q})(h, \boldsymbol{m})$ is a rotation of a cut through the swallowtail (cf. figure 1). The edge of the cusp corresponds to the critical condition (A 3). This generalizes the SQR theory for uniform flows to the case where $\boldsymbol{Q} = (Q_1, Q_1)$ has two components (cf. Benjamin & Lighthill 1954; Sewell & Porter 1980, § 4).

Appendix B. The shallow water equations

The classical equations governing flow in shallow water in two space dimensions when the vertical accelerations are negligible are (cf. Whitham 1974, p. 455)

$$h_t + (uh)_x + (vh)_y = 0, \quad u_t + uu_x + vu_y + gh_x = 0, \quad v_t + uv_x + vv_y + gh_y = 0. \tag{B 1}$$

The purpose of this appendix is two-fold: first, we show that the variational principle of Appendix A recovers all the features of the classical theory of uniform flows in two space dimensions and, second, we compute the eigenvalues of (B 1) linearized about the uniform state for comparison with the mean-flow perturbations that arise in the linear stability theory in § 5.

Supposing the flow in the horizontal plane to be irrotational, we can take $u = \phi_x$ and $v = \phi_y$, in which case the second and third equations of (B 1) combine into

$$\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) + gh = R(t), \tag{B 2}$$

where $R(t)$ is the Bernoulli function. Therefore, the set of equations (B 1) has the equivalent representation

$$\phi_t + u\phi_x + v\phi_y = \tfrac{1}{2}(u^2 + v^2) - gh + R(t),$$

$$h\phi_x = hu, \quad h\phi_y = hv, \quad -h_t - uh_x - hu_x - vh_y - hv_y = 0,$$

where, by hypothesis, $h > 0$. The above equations have the following structure:

$$\boldsymbol{J}_1 W_t + \boldsymbol{J}_2(W) W_x + \boldsymbol{J}_3(W) W_y = \nabla S(W) \tag{B 3}$$

where

$$\boldsymbol{J}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{J}_2(W) = \begin{pmatrix} 0 & 0 & 0 & u \\ 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 \\ -u & -h & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{J}_3(W) = \begin{pmatrix} 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h \\ -v & 0 & -h & 0 \end{pmatrix},$$

$$W \stackrel{\text{def}}{=} \begin{pmatrix} h \\ u \\ v \\ \phi \end{pmatrix} \quad \text{and} \quad S(W) = \tfrac{1}{2}h(u^2+v^2) - \tfrac{1}{2}gh^2 + R(t)h.$$

Note that, using the standard inner product on $\boldsymbol{R}^4$, without integration over x, y or t, it follows that

$$\nabla S(W) \stackrel{\text{def}}{=} \begin{pmatrix} \partial S/\partial h \\ \partial S/\partial u \\ \partial S/\partial v \\ \partial S/\partial \phi \end{pmatrix} = \begin{pmatrix} \tfrac{1}{2}(u^2+v^2) - gh + R(t) \\ hu \\ hv \\ 0 \end{pmatrix}.$$

In one space dimension, the functional $S(W)$ is related to the flow force, but in two space dimensions the physical relevance of $S(W)$ is not clear. The operators $\boldsymbol{J}_i$, $i = 1, 2, 3$, are skew-symmetric and it is easily verified that they are closed. Therefore, the above representation shows that the classical equations (B 1) can be formulated as a Hamiltonian system on a multisymplectic structure. Moreover, as in Appendix A, uniform flows correspond to critical points of S on level sets of Q_1 and Q_2, where $Q_1 = uh$ and $Q_2 = vh$. To see this let

$$\mathcal{F}(W; c_1, c_2) = S(W) - c_1 Q_1(W) - c_2 Q_2(W).$$

Then $\partial\mathcal{F}/\partial W = 0$ results in

$$u = c_1, \quad v = c_2 \quad \text{and} \quad R = gh_0 + \tfrac{1}{2}(c_1^2 + c_2^2), \tag{B 4}$$

in agreement with the result obtained for the exact equations in Appendix A (noting the correspondence $\gamma = -R$).

Substituting (B 4) into S, $\boldsymbol{Q}$ and R results in

$$S = h_0(c_1^2 + c_2^2) + \tfrac{1}{2}gh_0^2, \quad Q_1 = c_1 h_0, \quad Q_2 = c_2 h_0, \quad R = gh_0 + \tfrac{1}{2}(c_1^2 + c_2^2), \tag{B 5}$$

in agreement with (A 4) of Appendix A.

Finally in this appendix, we compute the eigenvalues associated with the linearization of (B 1) (or equivalently (B 3)) about the uniform flow. Let

$$h \mapsto h_0 + h, \quad u \mapsto c_1 + u \quad \text{and} \quad v \mapsto c_2 + v.$$

Then, linearizing (B 3) about this state

$$\boldsymbol{J}_1^0 W_t + \boldsymbol{J}_2^0 W_x + \boldsymbol{J}_3^0 W_y = [D^2 S(W_0) - c_1 D^2 Q_1(W_0) - c_2 D^2 Q_2(W_0)]W, \qquad \text{(B 6)}$$

where the 0 superscript on $\boldsymbol{J}_i$, $i = 1, 2, 3$, indicates that $\boldsymbol{J}_i$ is evaluated at the uniform state (B 4). Letting

$$W = \mathrm{Re}\{\widehat{W} \mathrm{e}^{\mathrm{i}(p_1 x + p_2 y - \Omega t)}\}$$

in equation (B 6) results in the following eigenvalue problem for $(\Omega, \widehat{W})$:

$$\begin{pmatrix} g & 0 & 0 & -\mathrm{i}(\Omega - p_1 c_1 - p_2 c_2) \\ 0 & -h_0 & 0 & +\mathrm{i}h_0 p_1 \\ 0 & 0 & -h_0 & +\mathrm{i}h_0 p_2 \\ \mathrm{i}(\Omega - p_1 c_1 - p_2 c_2) & -\mathrm{i}h_0 p_1 & -\mathrm{i}h_0 p_2 & 0 \end{pmatrix} \begin{pmatrix} \widehat{W}_1 \\ \widehat{W}_2 \\ \widehat{W}_3 \\ \widehat{W}_4 \end{pmatrix} = 0.$$

Setting the determinant of the above 4×4 matrix to zero results in

$$\Omega = p_1 c_1 + p_2 c_2 \pm \sqrt{g h_0} (p_1^2 + p_2^2)^{1/2}. \qquad \text{(B 7)}$$

Appendix C. Expressions for $\mathcal{S}$, $\overline{A}$, $\overline{\boldsymbol{B}}$, $\overline{P}$ and $\overline{\boldsymbol{Q}}$ evaluated on a TW

With the Fourier expansion (5.1) for a weakly nonlinear capillary-gravity TW and the identities (5.2), the leading order expressions for the seven basic functionals are

$$\begin{aligned} \mathcal{S} = {} & \tfrac{1}{4} k^2 (1 - t^2) b_1^2 (h + a_0) + \tfrac{1}{8} k^2 (1 - t^2) b_1^2 (a_2 \cosh 2kh + \tfrac{1}{2} k a_1^2 \sinh 2kh) \\ & + \tfrac{1}{2} \boldsymbol{m} \cdot \boldsymbol{m} (h + a_0) + \tfrac{1}{2} k^2 \frac{a_1 b_1 b_2}{\cosh 2kh} + \frac{k^2 h}{\cosh^2 2kh} b_2^2 + \tfrac{1}{2} \boldsymbol{m} \cdot \boldsymbol{k} a_1 b_1 \\ & - \tfrac{1}{2} g (a_0^2 + \tfrac{1}{2} a_1^2 + \tfrac{1}{2} a_2^2) + \tfrac{1}{4} \sigma k^2 a_1^2 + \sigma k^2 a_2^2 - (\tfrac{3}{8})^2 \sigma k^4 a_1^4 + \cdots \end{aligned}$$

$$\overline{A} = -\tfrac{1}{2} a_1 b_1 - \tfrac{1}{4} r t k a_1 b_1 (2a_0 + a_2) - \tfrac{1}{16} k^2 b_1 a_1^3 - a_2 b_2 - \tfrac{1}{2} k t_2 b_2 a_1^2 + \cdots$$

$$\begin{aligned} \overline{\boldsymbol{B}} = {} & \tfrac{1}{2} \boldsymbol{m} a_1 b_1 + \frac{1}{4} \frac{\boldsymbol{k}}{k} (t + kh(1 - t^2)) b_1^2 + \tfrac{1}{4} r \boldsymbol{k} (2a_0 + a_2) b_1^2 + \tfrac{3}{8} t k \boldsymbol{k} a_1^2 b_1^2 \\ & + \boldsymbol{k} a_1 b_1 b_2 + \frac{1}{4} \frac{\boldsymbol{k}}{k} \left(\frac{4kh + \sinh 4kh}{\cosh^2 2kh} \right) b_2^2 + \sigma \boldsymbol{k} \left(\tfrac{1}{2} a_1^2 + 2a_2^2 - \tfrac{3}{16} k^2 a_1^4 \right) + \cdots \end{aligned}$$

$$\overline{P} = a_0$$

$$\begin{aligned} \overline{\boldsymbol{Q}} = {} & \boldsymbol{m}(h + a_0) \\ & + \tfrac{1}{2} \boldsymbol{k} a_1 b_1 + \boldsymbol{k} a_2 b_2 + \tfrac{1}{2} t_2 \boldsymbol{k} k a_1^2 b_2 \tfrac{1}{16} \boldsymbol{k} k^2 a_1^3 b_1 + \tfrac{1}{4} t \boldsymbol{k} k (a_2 + 2a_0) a_1 b_1 + \cdots, \end{aligned}$$

where $t = \tanh kh$, $t_2 = \tanh 2kh$ and

$$\boldsymbol{k} = (k_1, k_2), \quad \boldsymbol{m} = (m_1, m_1), \quad \boldsymbol{B} = (B_1, B_2), \quad \text{and} \quad \boldsymbol{Q} = (Q_1, Q_2).$$

Appendix D. Group velocity and its derivatives for linear capillary-gravity waves

In this appendix the vector-valued expression for the group velocity and aspects of its derivatives with respect to wavenumber are recorded for use in § 5. The frequency ω and wavenumbers (k_1, k_2) (with $k \stackrel{\text{def}}{=} \sqrt{k_1^2 + k_2^2}$) satisfy

$$\omega^2 = (gk + \sigma k^3) \tanh kh, \tag{D 1}$$

where g and σ are the coefficients of gravity and surface tension, respectively. The vector-valued group velocity is defined by

$$\boldsymbol{c_g} = (c_g^{(1)}, c_g^{(2)}), \quad \text{with } c_g^{(j)} \stackrel{\text{def}}{=} \frac{\partial \omega}{\partial k_j}. \tag{D 2}$$

Using the definition, one finds

$$\boldsymbol{c_g} = |\boldsymbol{c_g}|\boldsymbol{k}, \quad \text{with } |\boldsymbol{c_g}| = \frac{g}{2\omega k}((1+\widetilde{T})(t + k\dot{t}) + 2t\widetilde{T}), \tag{D 3}$$

where

$$\widetilde{T} = \frac{\sigma k^2}{g}, \quad t = \tanh kh, \quad \dot{t} = \frac{\partial t}{\partial k} = h(1-t^2) \quad \text{and} \quad \boldsymbol{k} = (k_1, k_2). \tag{D 4}$$

Define

$$\mathrm{Hess}_{\boldsymbol{k}}(\omega) = \begin{pmatrix} \dfrac{\partial^2 \omega}{\partial k_1^2} & \dfrac{\partial^2 \omega}{\partial k_1 \partial k_2} \\ \dfrac{\partial^2 \omega}{\partial k_2 \partial k_1} & \dfrac{\partial^2 \omega}{\partial k_2^2} \end{pmatrix}. \tag{D 5}$$

Then, differentiating (D 3) we find

$$\mathrm{Hess}_{\boldsymbol{k}}(\omega) = |\boldsymbol{c_g}| \left(\boldsymbol{I}_2 - \frac{\boldsymbol{k}\boldsymbol{k}^{\mathrm{T}}}{k^2} \right) + \widehat{H} \frac{\boldsymbol{k}\boldsymbol{k}^{\mathrm{T}}}{k^2}, \tag{D 6}$$

where $\boldsymbol{I}_2$ is the identity operator on $\boldsymbol{R}^2$ and

$$\widehat{H} = \frac{g}{2\omega k}(6t\widetilde{T} + 2k\dot{t}(1+3\widetilde{T}) + k^2\ddot{t}(1+\widetilde{T})) - \frac{k^2}{\omega}|\boldsymbol{c_g}|^2. \tag{D 7}$$

Note that $\boldsymbol{k}\boldsymbol{k}^{\mathrm{T}}/k^2$ and $\boldsymbol{I}_2 - \boldsymbol{k}\boldsymbol{k}^{\mathrm{T}}/k^2$ are projection operators in the $\boldsymbol{k}$ (respectively $\boldsymbol{k}^{\perp}$) directions; $\mathrm{Hess}_{\boldsymbol{k}}(\omega)$ has two components; one projecting in the $\boldsymbol{k}$ direction and the other in the $\boldsymbol{k}^{\perp}$ direction.

Another related quantity that appears in the action flux and is important for the stability analysis is

$$\boldsymbol{f} = \frac{1}{4}\left(\frac{\omega^2}{t^2 k^2}(t + k\dot{t}) + 2\sigma k \right) \frac{\boldsymbol{k}}{k}. \tag{D 8}$$

In the stability analysis of § 5, the Jacobian of $\boldsymbol{f}$ with respect to $\boldsymbol{k}$ is needed. After some algebra we find, with $\boldsymbol{f} = (f_1, f_2)$,

$$\mathrm{Jac}_{\boldsymbol{k}}(\boldsymbol{f}) \stackrel{\text{def}}{=} \begin{pmatrix} \dfrac{\partial f_1}{\partial k_1} & \dfrac{\partial f_1}{\partial k_2} \\ \dfrac{\partial f_2}{\partial k_1} & \dfrac{\partial f_2}{\partial k_2} \end{pmatrix} = \frac{1}{2tk}(\omega\, \mathrm{Hess}_{\boldsymbol{k}}(\omega) - 3|\boldsymbol{c_g}|^2 \boldsymbol{k}\boldsymbol{k}^{\mathrm{T}}) + \frac{2\sigma}{\omega}|\boldsymbol{c_g}|\boldsymbol{k}\boldsymbol{k}^{\mathrm{T}}. \tag{D 9}$$

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